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Time:- 10:00a.m. to 1:00p.m.

Degree:- 3 (H)

Chapter:- Hydrostatics

Topic:- Fluid Pressure Theory + Problems

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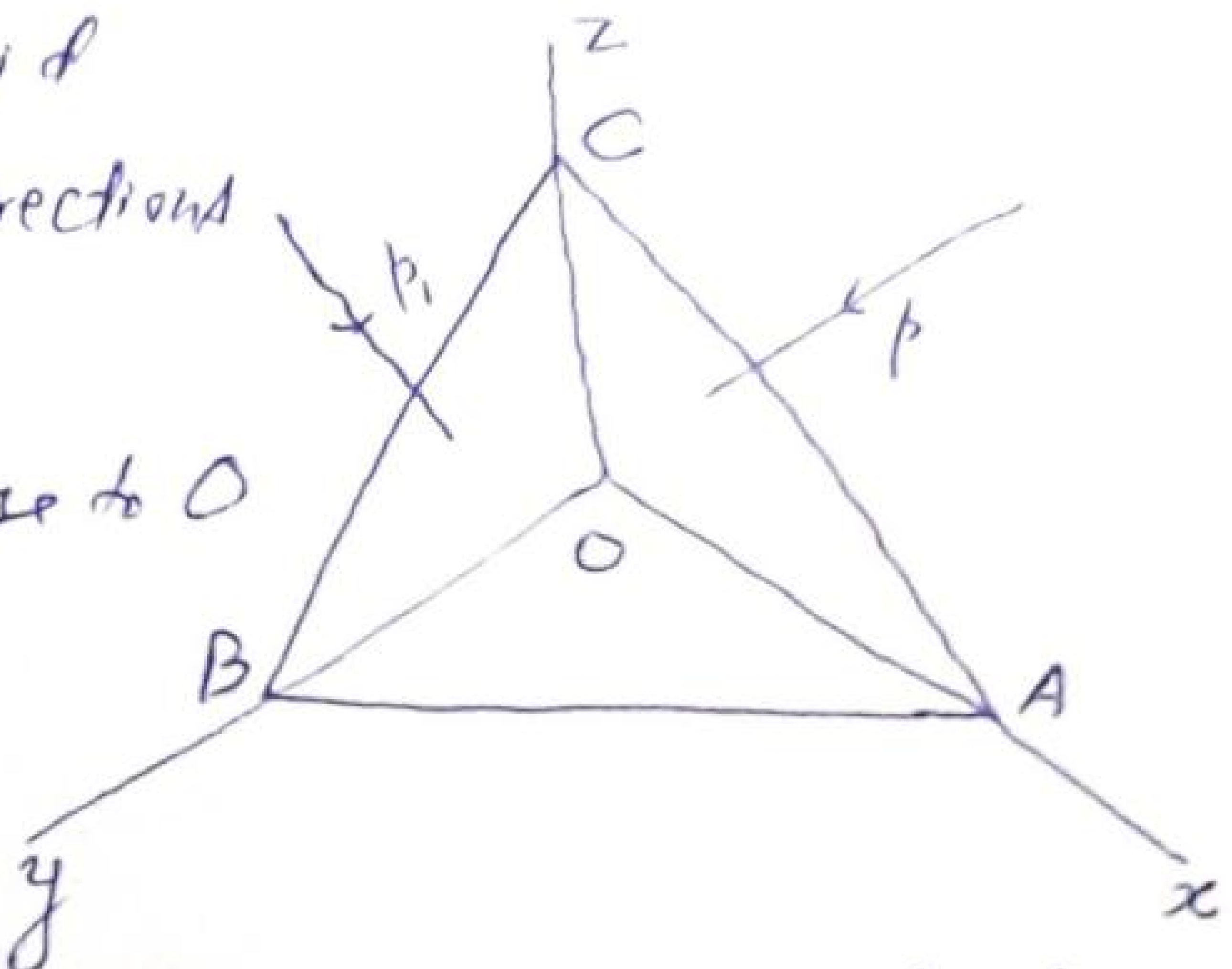
Hydrostatics

Topic: FLUID PRESSURE

(Art) Prove that the pressure at any point of a fluid at rest under gravity is the same in all directions.

Proof: Take any point O in the fluid and three mutually perpendicular directions O_x, O_y, O_z through it.

Taking points A, B, C close to O upon these lines respectively, draw the small tetrahedron OABC.



The portion of the fluid enclosed within this tetrahedron is at rest because the whole mass of the fluid is at rest.

Mass of the fluid within this tetrahedron

$= \rho \times$ Vol. of the tetrahedron, where ρ is the mean density of the fluid in it.

$= \rho \times \frac{1}{3}$ area of the base $\times$ height

$= \rho \times \frac{1}{3} \triangle BOC \times OA$

$= \frac{1}{6} \rho . OA . OB . OC$

Let p and p_1 be the mean pressure across the faces ABC and BOC respectively. The forces acting on the fluid within the tetrahedron are

(i) its weight $\frac{1}{6} g\rho . OA . OB . OC$ in direction making some angle θ with OA,

(ii) the thrust on the face OBC, $\frac{1}{2} . OB . OC . p_1$ along OA.

(iii) the thrust on the face ABC, p. area ABC, at right angles to ABC.

(iv) the thrusts on the faces AOB and AOC.

since the triangle BOC is the projection of the $\triangle ABC$ on the plane BOC, therefore

$\triangle BOC = \triangle ABC \cos\phi$, where ϕ is the angle between ABC and BOC, i.e. between OA and the perpendicular to ABC.

Resolving all the forces along OA, we get for equilibrium,

$$\frac{1}{2} OB.OC.p_1 - p \triangle ABC \cos\phi + \frac{1}{6}\rho g . OA.OB.OC \cos\theta = 0$$

$$\therefore \frac{1}{2} OB.OC.p_1 - p.\triangle BOC + \frac{1}{6}\rho g . OA.OB.OC.\cos\theta = 0$$

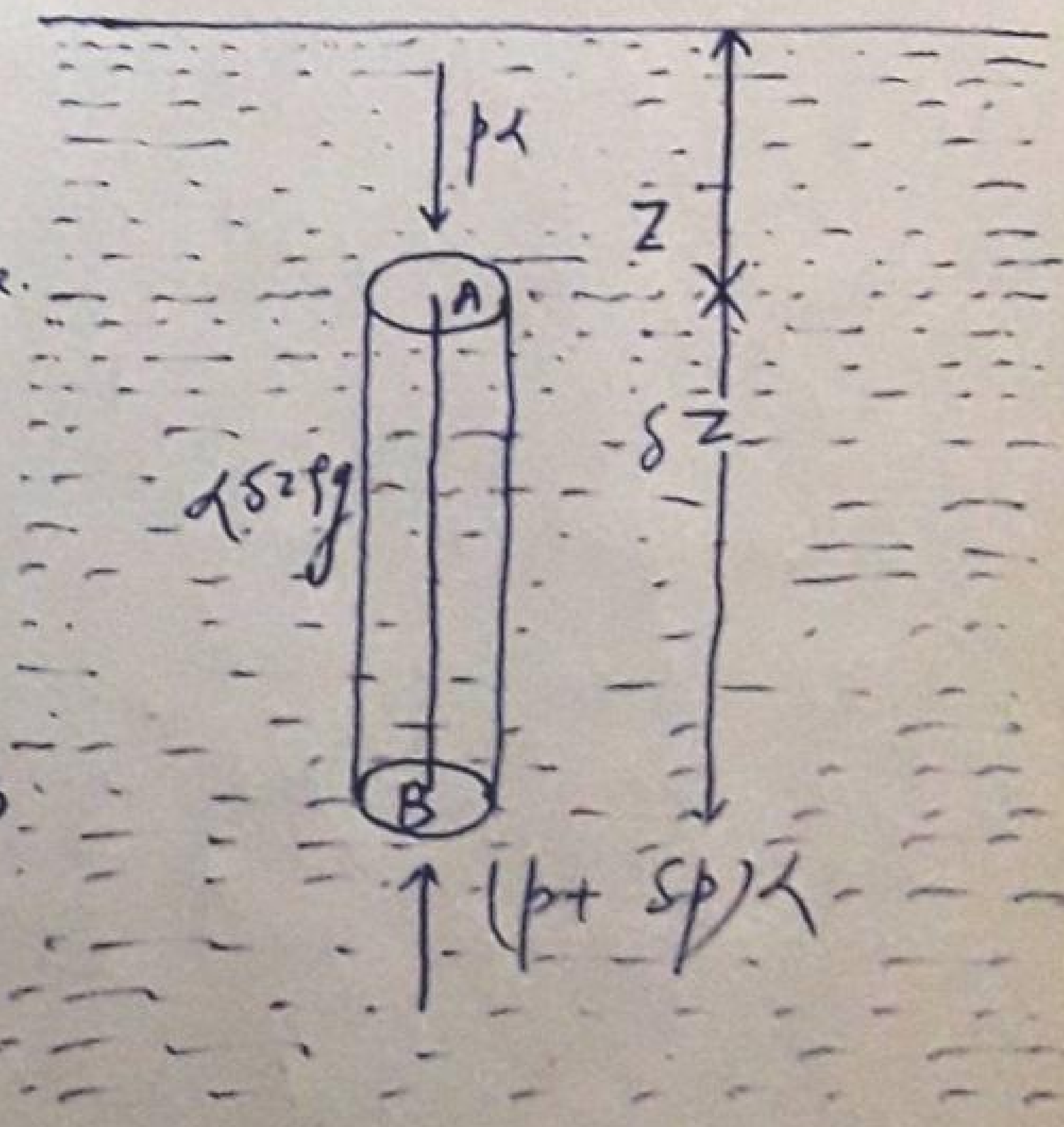
$$\therefore \frac{1}{2} OB.OC.p_1 - \frac{1}{2}p.OB.OC + \frac{1}{6}\rho g.OA.OB.OC.\cos\theta = 0$$

i.e. $$p_1 - p + \frac{1}{3}\rho g.OA.\cos\theta = 0 \quad \text{— (1)}$$

When OA, OB, OC are taken indefinitely small, then (1) reduces to $p_1 = p$ and the mean pressures on the faces become the pressures at O in the corresponding directions. Hence the theorem.

Art (i) Find the pressure at a depth z below the surface of a heavy homogeneous liquid, at rest under gravity, exposed to the pressure of the atmosphere.
(ii) Establish the formula $\frac{dp}{dz} = \rho g$.

Proof: Let A and B be two points on the same vertical in the liquid at depths z and $z + \delta z$ beneath the surface. Describe a cylinder of small cross-section α with AB as axis and with plane ends perpendicular to AB. Let the pressure at A and B be p and $(p+\delta p)$ respectively.



Let ρ be the density of the liquid

Consider the equilibrium of the liquid contained within this cylinder.
The forces acting on it are
(i) its weight, $\alpha\,\delta z\,\rho g$, acting vertically downwards,
(ii) the thrust, $p\alpha$ " " " on the end A.
(iii) the thrust $(p+\delta p)\alpha$ " " upwards on the end B
(iv) the thrusts on the curved surface, everywhere horizontal.

since the enclosed by this cylinder is in equilibrium obtain, by resolving the forces (acting on it) vertically.

$$p\alpha - (p+\delta p)\alpha + \alpha\,\delta z\,\rho g = 0$$

or $\quad \delta p = \rho g\,\delta z \quad$ or $\quad \dfrac{\delta p}{\delta z} = \rho g$

Making $\delta z \to 0$, we get $\dfrac{dp}{dz} = \rho g$ (pressure eqn)

or $dp = \rho g\,dz$

Integrating $p = \rho g z + K$. where K is the const. of integration

Let Π be the pressure of the atmosphere at the surface.

i.e. $\quad p = \Pi$ when $z = 0$.

$\therefore \quad \Pi = K$. Hence $p = \rho g z + K$.

This gives the pressure at a depth z in a homogeneous liquid contact with an atmosphere at rest

Note: If there is no atmospheric pressure (i.e. when $\Pi = 0$),

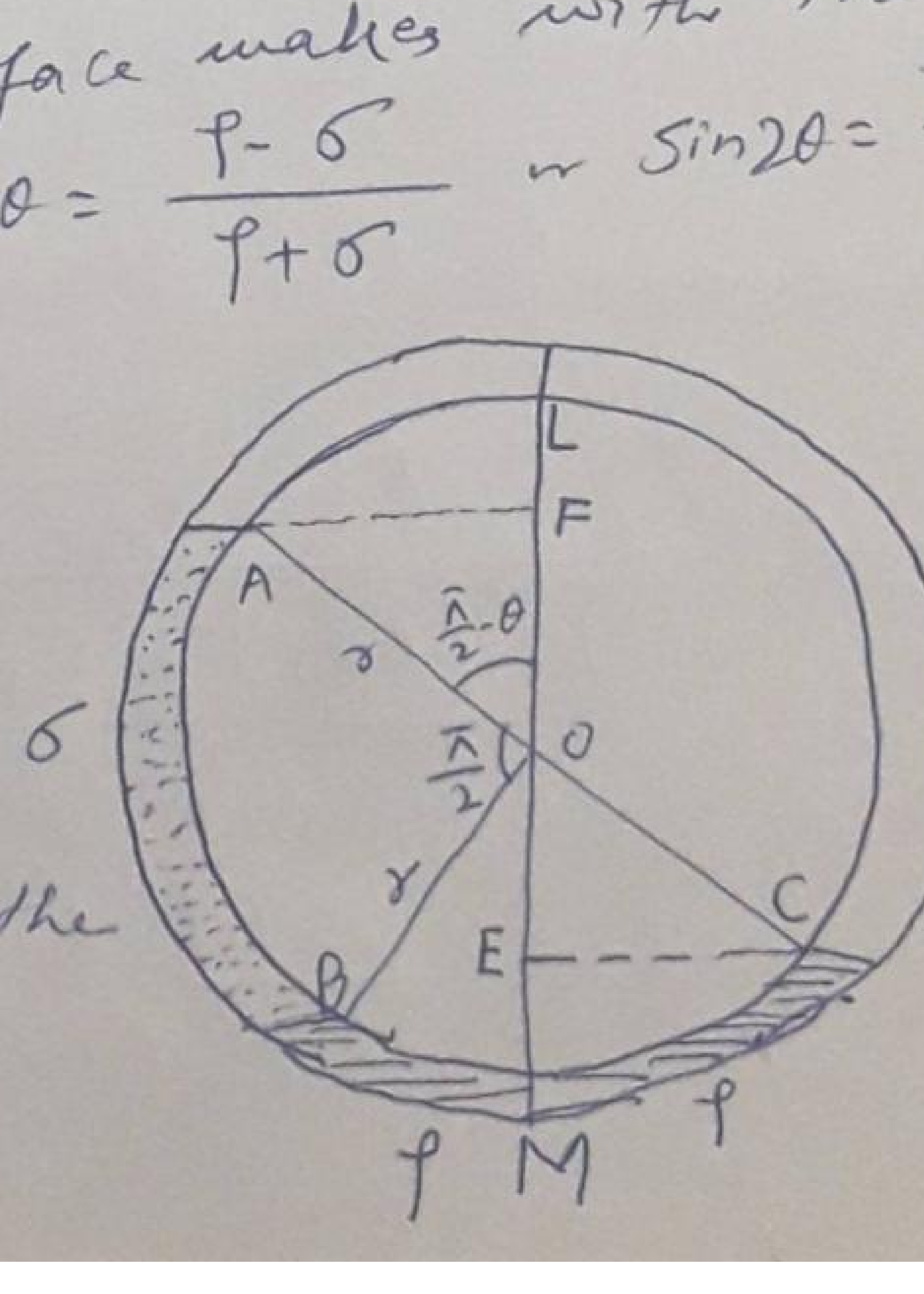
then $p = \rho g z$ i.e. $p \propto z$.

Hence the pressure at any point varies as the depth below the surface when there is no atmospheric pressure.

Problems

Ex-1 A small uniform tube is bent into the form of a circle whose plane is vertical. Equal quantities of two fluids of densities ρ and σ fill half the tube. Show that the radius passing through the common surface makes with the vertical an angle θ by given by $\tan\theta = \dfrac{\rho - \sigma}{\rho + \sigma}$ or $\sin 2\theta = \dfrac{\rho^2 - \sigma^2}{\rho^2 + \sigma^2}$

Soln:- Let AB and BC be the portions of the tube occupied by the fluids of densities σ and ρ respectively. Let the radius OB through the common surface make an angle θ with the vertical OM, M being the lowest point of the tube.



From question, the volume of the fluid in AB
= Vol of the fluid in BC.

$\therefore$ arc AB = arc BC

But the total arc occupied by the two fluid is half the Circumference of the tube.

$\therefore \angle AOB = \angle BOC = \frac{\pi}{2}$ and $\angle MOC = \frac{\pi}{2} - \theta = \angle AOL$

Draw AF, BD and CE perpendiculars on the vertical LM. If r the radius of the tube, then clearly

$OF = r\cos\left(\frac{\pi}{2}-\theta\right) = r\sin\theta, \quad OD = r\cos\theta,$

$OE = r\cos\left(\frac{\pi}{2}-\theta\right) = r\sin\theta$

Now the pressure at M due to the fluids on the side AM

= pressure due to fluid AB + pressure due to fluid BM

$= \sigma g \cdot FD + \rho g \cdot DM = \cancel{\sigma g (FD+}$

$= \rho g(OF+OD) + \rho g(OM-OD)$

$= \rho g(r\sin\theta + r\cos\theta) + \rho g(r - r\cos\theta)$

Again the pressure at M due to the fluid on the side CM

$= \rho g \cdot EM = \rho g(OM-OE) = \rho g(r - r\sin\theta)$

For equilibrium, these two pressures must be equal.

$\therefore \sigma g r(\sin\theta + \cos\theta) + \rho g r(1-\cos\theta) = \rho g r(1-\sin\theta)$

or $\sin\theta \cdot (\sigma+\rho) = \cos\theta(\rho-\sigma)$

$\Rightarrow \tan\theta = \frac{\rho-\sigma}{\rho+\sigma}$ ✓

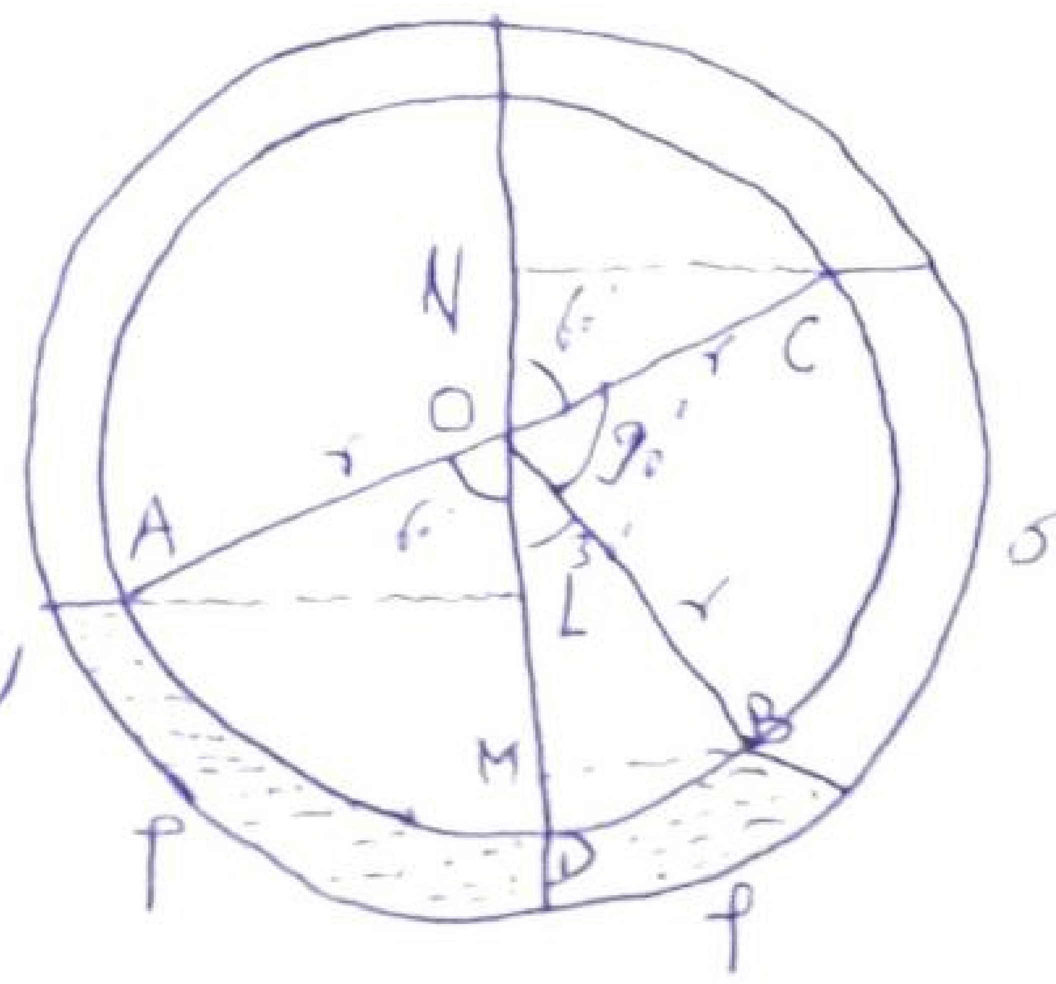
We know that $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\left(\frac{\rho-\sigma}{\rho+\sigma}\right)}{1+\left(\frac{\rho-\sigma}{\rho+\sigma}\right)^2}$

$\sin 2\theta = \frac{2\left(\frac{\rho-\sigma}{\rho+\sigma}\right)}{\frac{(\rho+\sigma)^2+(\rho-\sigma)^2}{(\rho+\sigma)^{\cancel{2}}}} = \frac{2(\rho-\sigma)(\rho+\sigma)}{\rho^2+\sigma^2+\cancel{2\rho\sigma}+\sigma^2+\rho^2-\cancel{2\sigma\rho}}$

$= \frac{\cancel{2}(\rho^2-\sigma^2)}{\cancel{2}(\rho^2+\sigma^2)}$ ✓

Ex-2 In a uniform circular tube two liquids are placed so as to subtend 90° each at the centre. If the diameter joining the two free surfaces be inclined at 60° to the vertical, prove that the densities of the two liquids are as $(\sqrt{3}+1) : (\sqrt{3}-1)$.

Soln:- Proof Let A and C be the surfaces of the two liquids of densities ρ and σ respectively and B their meeting pt.



So that $\angle AOB = \angle BOC = 90°$

The diameter AOC makes with the vertical OD an angle $AOD = 60°$

Draw AL, BM, and CN perpendiculars to the vertical diameter of which D is the lowest point. then clearly.

If r be the radius of the tube, then clearly

$OL = r\cos 60°$, $OM = r\cos 30°$, $ON = r\cos 60°$

The pressure at the point D due to the liquid in AD

$= \rho g . LD = \rho g(OD - OL) = \rho g(r - r\cos 60°)$

$= \rho g r(1 - \frac{1}{2}) = \frac{1}{2}\rho g r.$

The pressure at D due to the two liquids in DC

$= \rho g . MD + \sigma g . NM$

$= \rho g(OD - OM) + \sigma g(OM + ON)$

$= \rho g(r - r\cos 30°) + \sigma g(r\cos 30° + r\cos 60°)$

$= \rho g r(1 - \frac{\sqrt{3}}{2}) + \sigma g r(\frac{\sqrt{3}}{2} + \frac{1}{2})$

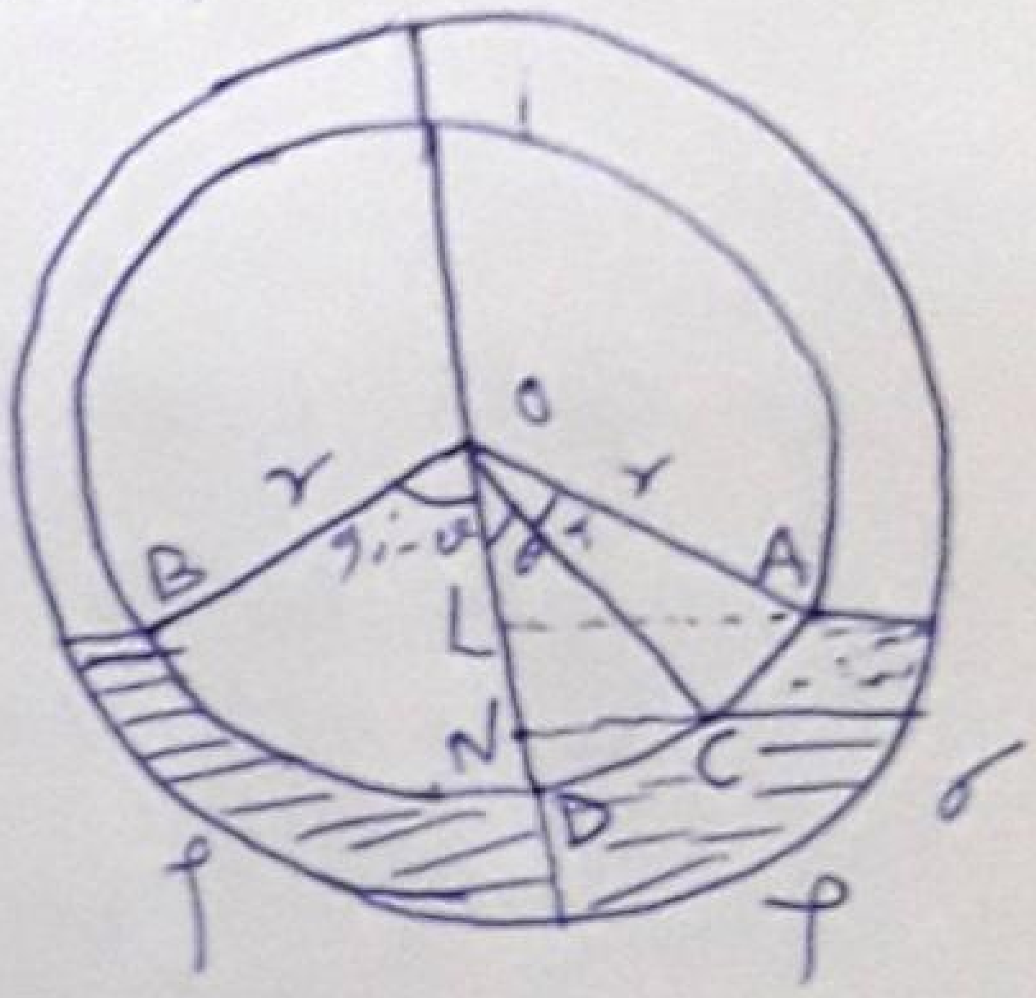
$= \frac{1}{2}\rho g r(2 - \sqrt{3}) + \frac{1}{2}\sigma g r(\sqrt{3}+1)$

For equilibrium, these two pressures must be equal.

$\therefore \frac{1}{2}\rho g r = \frac{1}{2}\rho g r(2 - \sqrt{3}) + \frac{1}{2}\sigma g r(\sqrt{3}+1)$

or $\rho(1 - 2 + \sqrt{3}) = \sigma(\sqrt{3}+1) \Rightarrow \frac{\rho}{\sigma} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

Ex-3 A fine circular tube in the vertical plane contains a column of liquid, of density ρ, which subtends a right angle at the centre and a column of density σ subtending an angle α. Prove that the radius through the common surface makes with the vertical an angle $\tan^{-1}\dfrac{\rho-\sigma+\sigma\cos\alpha}{\rho+\sigma\sin\alpha}$.



Sol:- Let O be the centre of the circular tube, the radius and D the lowest pt.

Let BC contain the liquid of density ρ and CA contain the liquid density ρ where $\angle BOC = 90°$ and $\angle COA = \alpha$

Let $\angle DOC = \theta$

Then $\angle BOD = 90-\theta$

Draw AL, BM, CN perpendiculars on OD.

If r be the radius of the tube, then $OL = r\cos(\theta+\alpha)$,
$OM = r\cos(90-\theta)$, $ON = r\cos\theta$,

Since D is the lowest point, therefore the pressure at D due to the liquids on both the sides of the tube must be equal.

$$\rho g.\,DM = \rho g.\,DN + \sigma g.\,NL$$

$$\text{or}\quad \rho(DM-DN) = \sigma NL$$

$$\text{or}\quad \rho.MN = \sigma.NL$$

$$\text{or}\quad \rho(ON-OM) = \sigma(ON-OL)$$

$$\text{or}\quad \rho(r\cos\theta - r\sin\theta) = \sigma[r\cos\theta - r\cos(\theta+\alpha)]$$

$$\text{or}\quad \rho(\cos\theta-\sin\theta) = \sigma(\cos\theta - \cos\theta.\cos\alpha + \sin\theta.\sin\alpha)$$

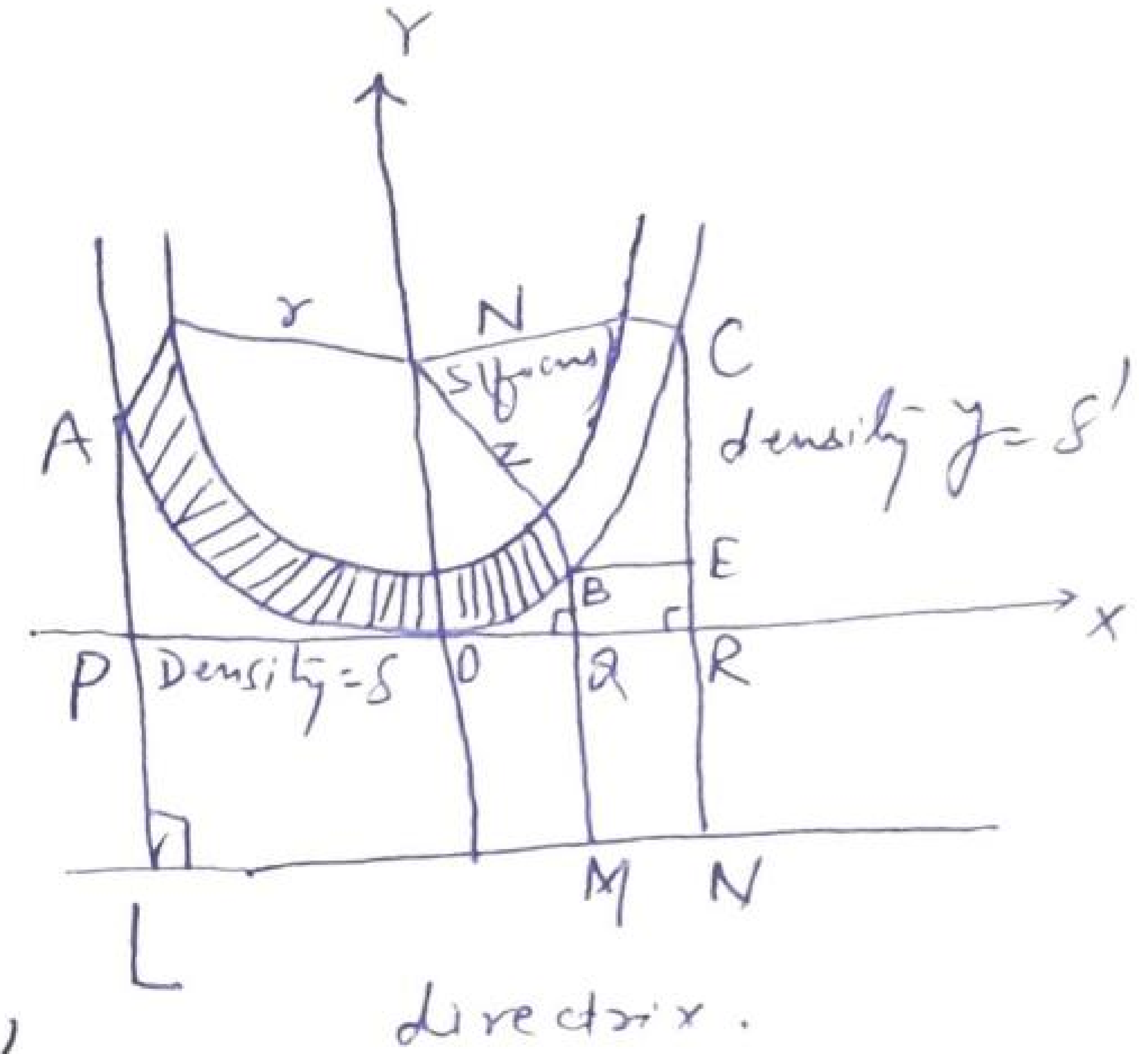
$$\text{or}\quad \sin\theta(\rho+\sigma\sin\alpha) = \cos\theta(\rho-\sigma+\sigma\cos\alpha)$$

$$\text{or}\quad \tan\theta = \frac{\rho-\sigma+\sigma\cos\alpha}{\rho+\sigma\sin\alpha}.$$

Hence the required angle is $\tan^{-1}\left(\dfrac{\rho-\sigma+\sigma\cos\alpha}{\rho+\sigma\sin\alpha}\right)$

Ex-4 A tube in the form of a parabola held with its vertex downwards and axis vertical, is filled with two different liquids of densities δ and δ'. If the distances of the free surfaces of the liquids from the focus be r and r' respectively, show that the distance of their common surface from the focus is $\frac{r\delta - r'\delta'}{\delta - \delta'}$.

Soln:- Let O be the vertex, LN the directrix and S the focus of the parabola.



The liquids of densities δ and δ' occupy the portions AB and BC of the tube. B is the common surface

Join A, B, C from S.

Then $AS = r$, $CS = r'$, $BS = Z$ (say).

We have to prove that $Z = \dfrac{r\delta - r'\delta'}{\delta - \delta'}$

From A, B, C draw perpendiculars AL, BM, CN on the directrix cutting OX at P, Q, R respectively.

Then, by the defn of parabola, the focal distance of any point on the parabola = the distance of the point from the directrix.

$\therefore SL = AL = r$

$SB = BM = z$,

$CS = CN = r'$.

The pressure at O due to the liquid in the portion OA of the tube $= \rho g AP = \delta g (AL - PL) = \delta g (r - OD) = \delta g (r - b)$, where $b = OD$.

Again, the pressure at O due to the liquid in the portion OC of the tube

$= \delta g BQ + \delta' g CE$, where $BE \perp CN$

$= \delta g (BM - QM) + \delta' g (CN - EN)$

$= \delta g(z - OD) + \delta' g(\gamma' - BM)$

$= \delta g(z - b) + \delta' g(\gamma' - z)$

For equilibrium, these two pressures must be equal.

$\therefore \delta g(\gamma - b) = \delta g(z - b) + \delta' g(\gamma' - z)$

or $\gamma\delta - b\delta = z\delta - b\delta + \gamma'\delta' - z\delta'$

or $\gamma\delta - \gamma'\delta' = z(\delta - \delta')$ $\Rightarrow z = \dfrac{\gamma\delta - \gamma'\delta'}{\delta - \delta'}$