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B.Sc Paper III
GrB.

Scattering of electromagnetic wave.

(a) Scattering by free charge or Thomson's scattering.

Let us consider an electron of mass $m$ and charge "$e$" in the path of Polarised light.

Let the incident E.M wave having electric intensity $E$ along $x$-direction moving along $z$-direction. The electromagnetic force is $e.E$.

The equation of motion of electron is given by

$$m.\frac{d^2x}{dt^2} = eE = eE_0 e^{-i(\omega t-Kx)} \quad \text{(A)}$$

or, $$\frac{d^2x}{dt^2} = \frac{eE_0}{m} e^{-i(\omega t-Kx)} \quad \text{(B)}$$

$$\Rightarrow x = -\frac{eE_0}{\omega^2 m} e^{-i(\omega t-Kx)}$$

The oscillating charge behaves like a dipole having dipole moment

$$p = ex = -\frac{e^2E_0}{\omega^2 m} e^{-i(\omega t-Kx)}$$

or $$p = p_0 e^{-i(\omega t-Kx)}$$

where $p_0 = -\dfrac{e^2E_0}{m\omega^2}$

The oscillating dipole radiates energy. The average energy radiating per second per unit area is given by

$$\omega_s = \frac{1}{4\pi\epsilon_0}\frac{\omega^4 p_0^2}{8\pi c^3 r^2}\sin^2\alpha.$$

$$= \frac{1}{4\pi\epsilon_0}\frac{e^4E_0^2}{8\pi c^3 r^2 m^2}\sin^2\alpha \quad \text{(C)}$$

For plane electromagnetic wave the average incident energy is given by

$$\omega_i = \frac{1}{2}c\epsilon_0 E_0^2$$

The differential scattering cross-section is given by

$$\sigma(\theta) = \frac{\omega_s}{\omega_i} r^2$$

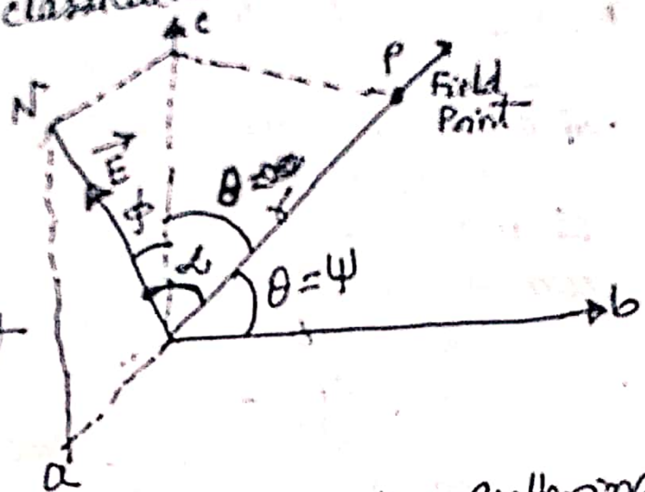
$$= \frac{1}{4\pi\epsilon_0}\frac{e^4E_0^2\sin^2\alpha}{8\pi c^3 r^2 m^2}\cdot\frac{2}{c\epsilon_0 E_0^2} r^2$$

or, $\sigma(\theta) = \left(\frac{e^2}{4\pi m c^2 \epsilon_0}\right)^2 \sin^2\alpha$ (2)

or, $\sigma(\theta) = r_0^2 \sin^2\alpha$.

where $r_0 = \frac{e^2}{4\pi m c^2 \epsilon_0}$ = classical radius of the electron.

Incident polarised light




Let $\phi$ is the polaring angle and $\theta$ is the scattering angle of e.m. wave and $\alpha$ is the angle between position vector r & electric vector E

Now $ON = r\cos\alpha$ & $OM = r\sin\theta$

$\therefore ON = r\cos\alpha = OM\cos\phi = r\sin\theta\cos\phi$.

Again $\cos^2\alpha = \sin^2\theta \cdot \cos^2\phi$.

$\therefore 1 - \sin^2\alpha = (1-\cos^2\theta)\cos^2\phi$.

or, $\sin^2\alpha = 1 - \cos^2\phi(1-\cos^2\theta)$

For plane polarised light, $\phi = 0$

$\therefore \sin^2\alpha = \cos^2\theta$

For unpolarised light, the average value of $\phi$ is equal to $\sqrt{1/2}$

$\therefore \sin^2\alpha = 1 - \frac{1}{2}(1-\cos^2\theta) = \frac{1}{2}(1+\cos^2\theta)$

$= \frac{1}{2}(1+\cos^2\theta)$

The scattering cross-section area is given by

$$\boxed{\sigma(\theta) = r_0^2 \frac{1}{2}(1+\cos^2\theta)}$$

This is thomson scattering formula.

(B) Scattering by bounded charge (Rayleigh scattering)

Let us consider following forces are acting upon the bounded electrons.

(i) Inertial force $= m\cdot\frac{d^2x}{dt^2}$

(ii) Frictional force which is proportional to velocity

$$f_r = \alpha\frac{dx}{dt}$$

(iii) Damping force which is proportional to the displacement $f_t = Kx$

(iv) The periodic electromagnetic force.

$$F = eE_0 e^{i(\omega t - Kx)}$$

The eqn of motion is given by.

$$m\cdot\frac{d^2x}{dt^2} + \alpha\cdot\frac{dx}{dt} + Kx = eE_0 e^{-i(\omega t - Kx)}$$

or, $$\frac{d^2x}{dt^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{K}{m}x = \frac{e}{m}E_0 e^{-i(\omega t - Kx)} \quad \text{—— (A)}$$

or, $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{e}{m}E_0 e^{-i(\omega t - Kx)}$$

where $\gamma = \alpha/m$ & $\omega_0^2 = K/m$.

The solution of complementary function is given by.

$$\frac{d^2x}{dt^2} + \gamma\cdot\frac{dx}{dt} + \omega_0^2 x = 0 \quad \text{—— (B)}$$

Let $x = Ae^{\alpha t}$, the eqn 'B' may be written as

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0$$

$$\therefore \alpha = -\frac{\gamma}{2} \pm \sqrt{\gamma^2/4 - \omega_0^2}$$

$$= -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \gamma^2/4}$$

$$\therefore x = e^{-\gamma t/2}\left[A_1 e^{+i\sqrt{(\omega^2 + \gamma^2/4)}\,t} + A_2 e^{-\left(i\sqrt{\omega^2 - \gamma^2/4}\right)t}\right] \quad \text{—— (C)}$$

The factor decreases exponentially which die after some time. The soln. of Perpendicular integral is given by

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2(x) = e/m\,E_0 e^{-i(\omega t - Kx)} \quad \text{—— (D)}$$

Let $x = B e^{-i(\omega t - kx)}$

From eqn (2) we get

$$B = \frac{eE_0}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

$$\therefore x = \frac{eE_0}{m(\omega_0^2 - \omega^2 - i\gamma\omega)} e^{-i(\omega t - kx)}$$

$$= \frac{eE_0\left[(\omega_0^2 - \omega^2) + i\gamma\omega\right]}{m\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]} e^{-i(\omega t - kx)}$$

$$= \frac{eE_0}{m} \frac{e^{-i(\omega t - kx)}}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}} \left[\frac{\omega_0^2 - \omega^2}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}} + \frac{i\gamma\omega}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}}\right]$$

$$= \frac{eE_0}{m} \frac{e^{-i(\omega t - kx)}}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}} \left[\cos\theta - i\sin\theta\right]$$

$$= \frac{eE_0}{m} \frac{e^{-i(\omega t - kx) + \delta)}}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}} \quad \text{——— (E)}$$

where $\delta = \tan^{-1} \dfrac{\gamma\omega}{\omega_0^2 - \omega^2}$

The comple solution of equation (A) is given by

$$x = e^{-\gamma t/2}\left[A_1 e^{i\beta t} + A_2 e^{-i\beta t}\right] + \frac{eE_0}{m} \frac{e^{i(\omega t - kx + \delta)}}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}} \quad \text{— (F)}$$

The first term may be neglected due to -ve exponential

The general eqn of oscillating charge due to dipole moment $p = ex$

$$= \frac{e^2 E_0}{m\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}} \left[e^{+i(\omega t - kx + \delta)}\right].$$

$$\text{or, } p = p_0 e^{-i(\omega t - kx - \delta)} \quad \text{——— (G)}$$

$$\text{where } p_0 = \frac{e^2 E}{m\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}} \quad \text{——— (H)}$$

The average energy radiated per second per unit area is given by.

(9)

$$S = \frac{1}{4\pi\epsilon_0}\frac{\omega^4}{8\pi c^2 r^2}\cdot p_0^2 \sin^2\theta$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{\omega^4 p^2}{8\pi c^2 r^2}\frac{e^4 E_0^2 \sin^2\theta}{m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]} \quad \text{— (11)}$$

The average incident radiation (Poynting Vector)

$$S_{in} = \frac{1}{2}\epsilon_0 c E_0^2$$

The differential scattering crossection is given by

$$\sigma(\theta) = \frac{S}{S_{in}}\gamma^2$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\omega^4}{8\pi c^2 r^2}\frac{e^2 E_0^2 \sin^2\theta\, r^2}{m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]\, c\epsilon_0 E_0^2} \quad \text{— (I)}$$

On solving we get

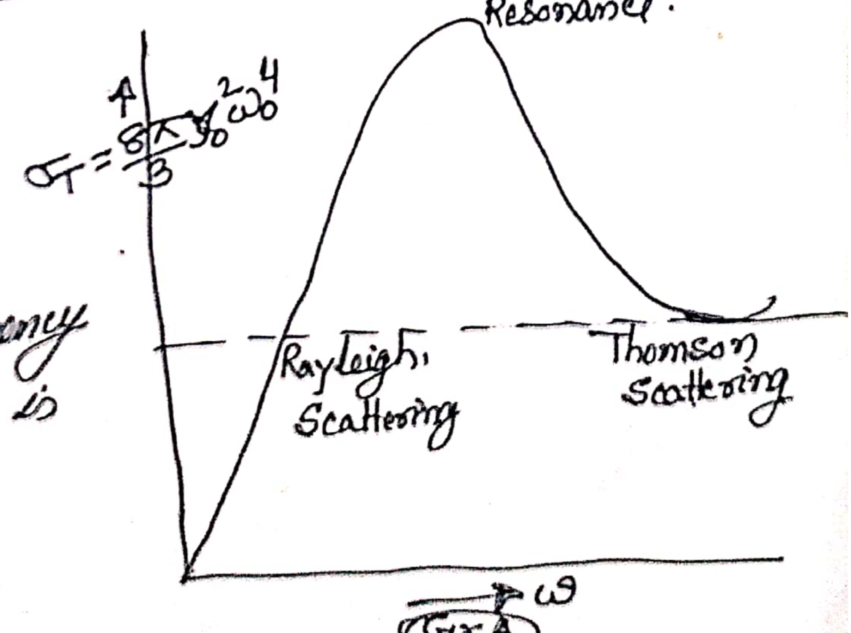
$$\sigma(\theta) = \gamma_0^2\omega^4\left(\frac{1+\cos^2\theta}{2}\right)\Big/(\omega_0^2-\omega^2)^2+\gamma^2\omega^2$$

The total scattering crossection is given by

$$\sigma_T = \int_0^\pi \frac{\gamma_0^2\omega^4}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}\left(\frac{1+\cos^2\theta}{2}\right)\cdot 2\pi\sin\theta\, d\theta \quad \text{— J}$$

$$\boxed{\sigma_T = \frac{8\pi}{3}\frac{\gamma_0^2\omega^4}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}}$$

The eqn I represents the total scattering crossection for elastically bound electron. The scattering crossection is a function of frequency of incident radiation is given by GrapA.




Gr A