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Chapter:- Area of curves length determined from Polar Equation

Topic:- Integral Calculus

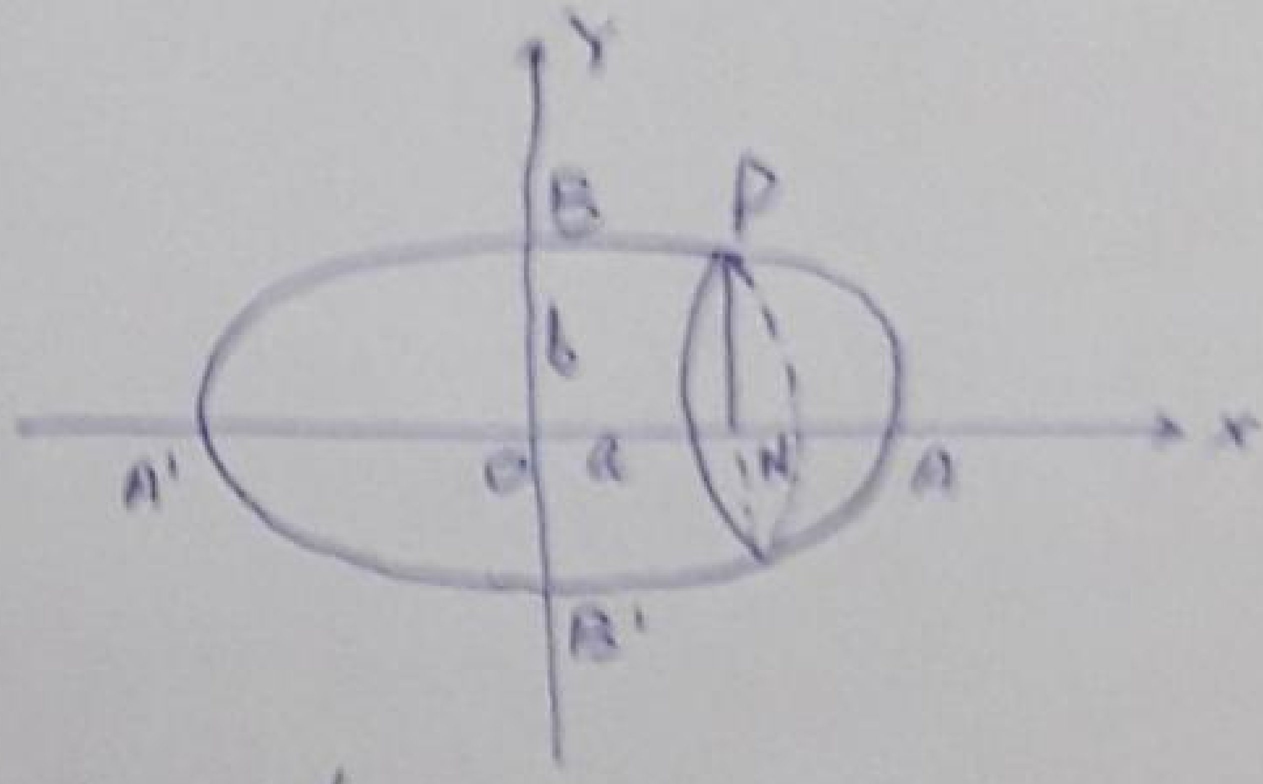
By
Professor
(Dr.) Mohammad
Eqbalu zafar

Volumes and Surface Areas of Solids of Revolution

Ex-1

Find the volume and surface area of the solid formed by the revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis.

Soln: $V = \int_{-a}^{a} \pi y^2 \, dx$



$$= \pi \int_{-a}^{a} b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \pi \int_{-a}^{a} \frac{b^2(a^2-x^2)}{a^2} dx = \frac{\pi b^2}{a^2} \int_{-a}^{a} (a^2 - x^2)\, dx$$

$$= \frac{\pi b^2}{a^2} \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{\pi b^2}{a^2} \left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right]$$

$$= \frac{\pi b^2}{a^2} \cdot 2\left(a^3 - \frac{a^3}{3}\right) = \frac{\pi b^2}{a^2} \times 2 \times \frac{2a^3}{3} = \frac{4}{3} \pi a b^2$$

Again, the surface area of the solid $S = \int 2\pi y \, ds$
where the limits of x are from $x = -a$ to $x = +a$.

Now, $x = a\cos\theta$ and $y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = b\cos\theta$$

Thus $\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\sin^2\theta + b^2\cos^2\theta$

$$\Rightarrow ds = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$

Also, $\left.\begin{array}{l} x = a \Rightarrow \theta = 0 \\ x = -a \Rightarrow \theta = \pi \end{array}\right\}$

Hence $S = 2\pi \int_0^{\pi} (b\sin\theta)\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta = 4\pi \int_0^{\pi/2} (b\sin\theta)\sqrt{a^2 - a^2e^2\cos^2\theta}\, d\theta$

$\boxed{\text{since } b^2 = a^2(1-e^2)}$

$$= 4\pi ab \int_0^{\pi/2} \sqrt{1 - e^2\cos^2\theta}\, \sin\theta \, d\theta$$

Put $e\cos\theta = u \Rightarrow du = -e\sin\theta\, d\theta$.

Also $\theta = 0 \Rightarrow u = e$ and $\theta = \frac{\pi}{2} \Rightarrow u = 0$.

$$\therefore S = 4\pi ab \int_e^0 \sqrt{1-u^2} \times \left(\frac{-du}{e}\right)$$

$$= \frac{4\pi ab}{e} \int_0^e \sqrt{1-u^2}\, du = \frac{4\pi ab}{e} \left[\frac{u\sqrt{1-u^2}}{2} + \frac{1}{2}\sin^{-1}(u)\right]_0^e$$

$$= \frac{2\pi ab}{e}\left[e\sqrt{1-e^2} + \sin^{-1}e\right] = 2\pi ab\left[\sqrt{1-e^2}+\frac{1}{e}\sin^{-1}e\right]$$

Ex-2 Find the volume of the solid formed by the revolution of the loop of the curve $y^2 = \frac{x^2(a-x)}{a+x}$ about the x-axis.

Sol^n:- (i) The curve passes through the origin.

(ii) It is symmetrical about the x-axis

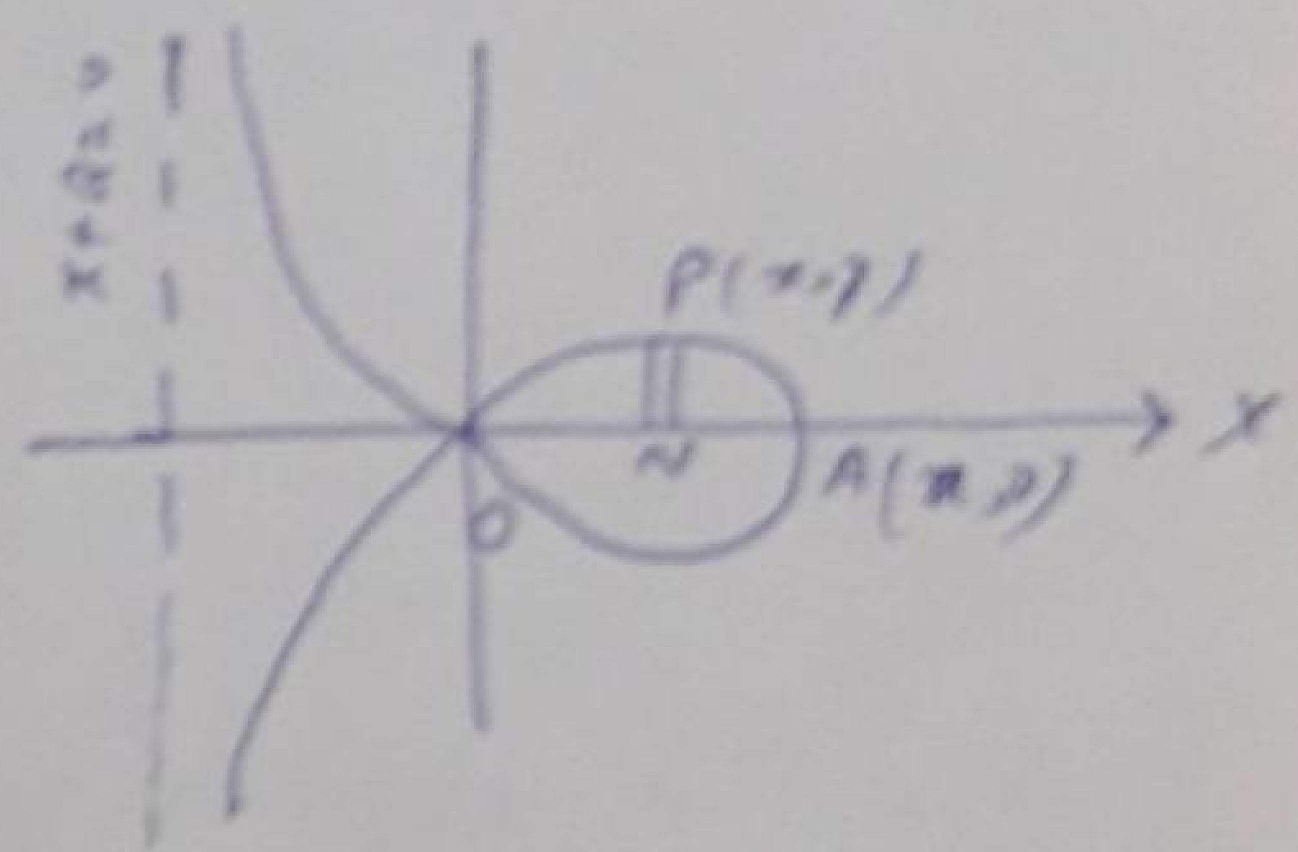
(iii) Put $y=0 \Rightarrow x=0, a$ and put $x=0$, $y=0$

(iv) when $x>a \Rightarrow y^2$ is $-$ive and so y is imaginary.
Thus the loop lies between $x=0$ to $x=a$

(v) Eq^n of the asymptote is $x+a=0 \Rightarrow x=-a$.

Required volume,

$$V = \int_0^a \pi y^2\,dx = \pi\int_0^a \frac{x^2(a-x)}{a+x}\,dx \quad \text{— (1)}$$



Put $a+x = z \Rightarrow dx = dz$

$a-x = a-z+a = 2a-z$.

Also $x=0 \Rightarrow z=a$ and $x=a \Rightarrow z=2a$.

From (1), $$V = \pi\int_a^{2a}(z-a)^2\cdot\frac{2a-z}{z}\,dz = \pi\int_a^{2a}\frac{(z^2-2az+a^2)(2a-z)}{z}\,dz$$

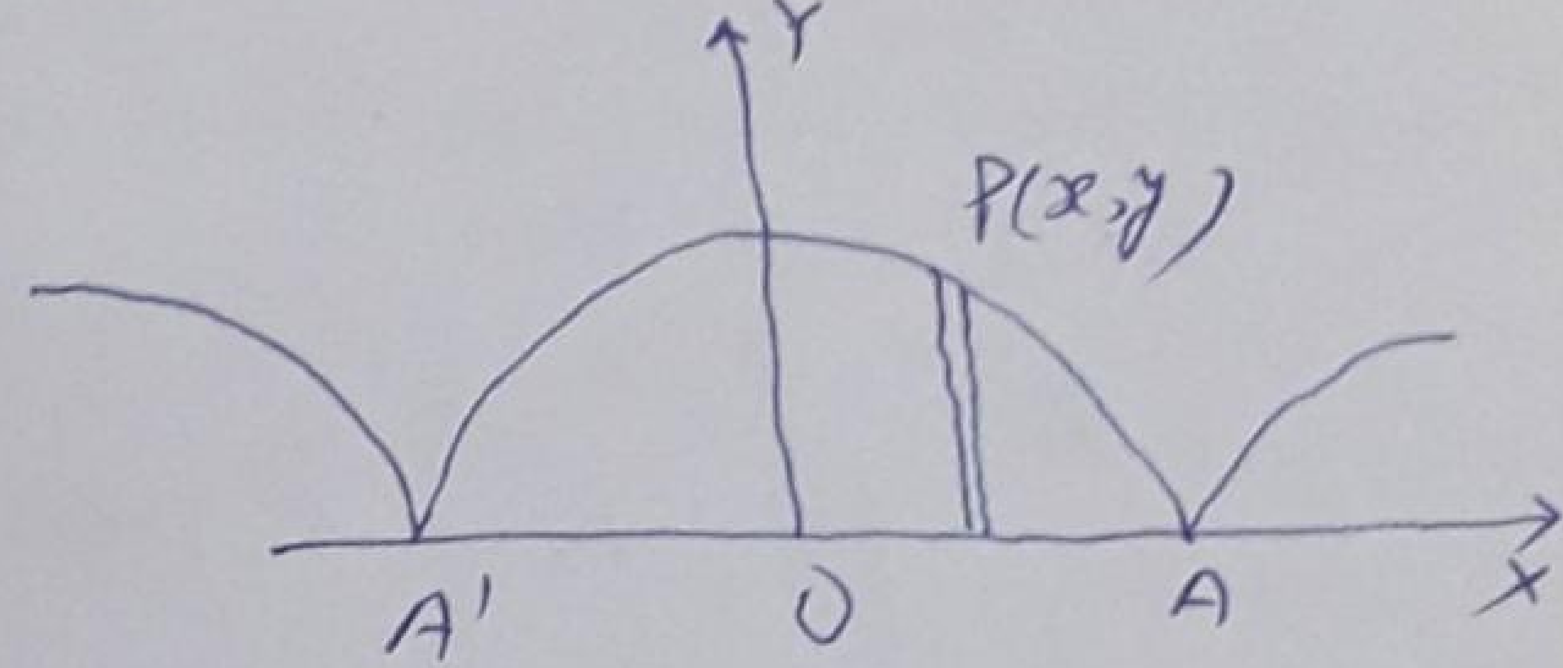
$$V = \pi\int_a^{2a}\left(4az-5a^2-z^2+\frac{2a^3}{z}\right)dz$$

$$= \pi\left[\left\{2a\cdot4a^2 - 5a^2\cdot2a - \frac{1}{3}8a^3 + 2a^3\log 2a\right\} - \left\{2a\cdot a^2 - 5a^3 - \frac{1}{3}a^3 + 2a^3\log a\right\}\right]$$

$$= \pi\left[\left(\frac{10}{3}-\frac{14}{3}\right)a^3 + 2a^3\{\log(2a)-\log a\}\right]$$

$$= \pi\left(2a^3\log 2 - \frac{4}{3}a^3\right) = 2\pi a^3\left[\log 2 - \frac{2}{3}\right]$$

Ex-3 Find the volume and surface area of the solid generated by revolving the cycloid $\left.\begin{array}{l} x = a(\theta + \sin\theta) \\ y = a(1+\cos\theta) \end{array}\right\}$ about its base.



Required volume, $V = 2\int \pi y^2 dx$

$$V = 2\pi \int_0^{\pi} a^2 (1+\cos\theta)^2 \cdot a(1+\cos\theta)\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} (1+\cos\theta)^3 d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^3 d\theta = 16\pi a^3 \int_0^{\pi} \cos^6\frac{\theta}{2}\, d\theta = 16\pi a^3 \int_0^{\pi/2} \cos^6\phi \cdot 2\, d\phi$$

putting $\phi = \frac{\theta}{2}$.

$$= 32\pi a^3 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 5\pi^2 a^3$$

Required surface, $S = 2\int 2\pi y\, ds$ —— (1)

But $\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$

$$= a^2(1+\cos\theta)^2 + a^2(-\sin\theta)^2$$

$$= a^2(2 + 2\cos\theta) = 2a^2 \cdot 2\cos^2\frac{\theta}{2} = 4a^2\cos^2\frac{\theta}{2}$$

$$\Rightarrow \frac{ds}{d\theta} = 2a\cos\frac{\theta}{2}.$$

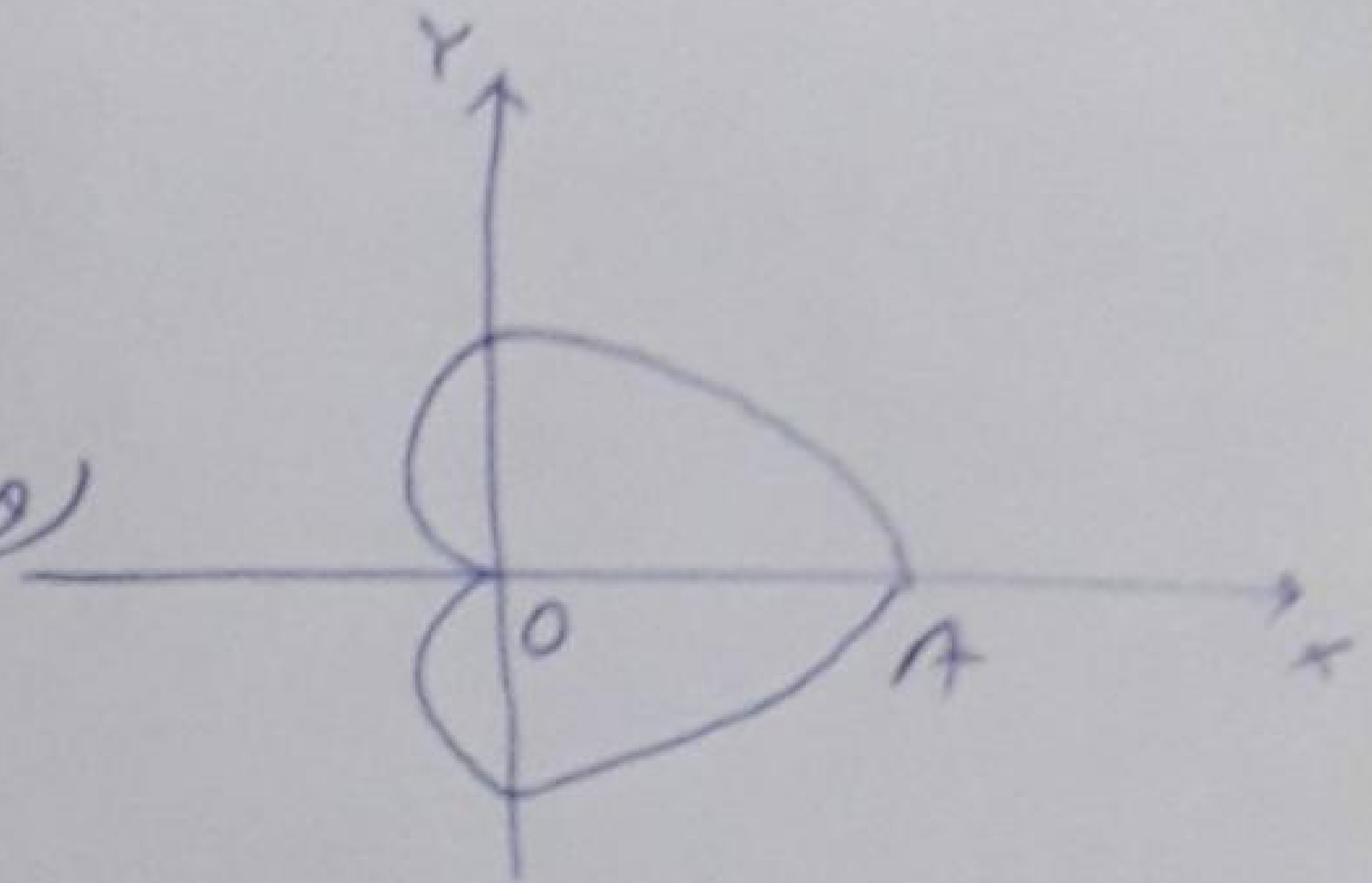
Hence, from (1) $S = 4\pi \int_0^{\pi} a(1+\cos\theta) \cdot 2a\cos\frac{\theta}{2}\, d\theta$

$$S = 8\pi a^2 \int_0^{\pi} 2\cos^2\frac{\theta}{2} \cdot \cos\frac{\theta}{2}\, d\theta = 16\pi a^2 \int_0^{\pi} \cos^3\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi/2} \cos^3\phi \cdot 2\, d\phi \quad \left[\because \frac{\theta}{2} = \phi\right]$$

$$= 32\pi a^2 \cdot \frac{2}{3} = \frac{64}{3}\pi a^2$$

Ex-1 The cardiode $r = a(1+\cos\theta)$ revolves about the initial line. Find the surface area and volume of the figure formed.



Soln:- The curve is symmetrical about the initial line.

Required volume

$$V = \int \pi y^2 dx = \pi \int (r\sin\theta)^2 d(r\cos\theta)$$

$$= \pi \int r^2 \sin^2\theta \, d(r\cos\theta)$$

$$= \pi \int a^2(1+\cos\theta)^2 \sin^2\theta \, d\{a(1+\cos\theta)\cos\theta\}$$

$$= \pi a^3 \int_{\pi}^{0} (1+\cos\theta)^2 \sin^2\theta \{-\sin\theta - 2\sin\theta\cos\theta\} d\theta.$$

[The limits of θ are from π to 0 because x increases as θ decreases from π to 0]

$$= -\pi a^3 \int_{\pi}^{0} (1+\cos\theta)^2 (1-\cos^2\theta)(1+2\cos\theta)\sin\theta \, d\theta \quad \text{— (1)}$$

Put $\cos\theta = u \Rightarrow -\sin\theta \, d\theta = du$.

Also $\theta = \pi \Rightarrow u = -1$ and $\theta = 0 \Rightarrow u = 1$.

Thus (1) becomes $= \pi a^3 \int_{-1}^{1} (1+u^2)(1-u^2)(1+2u)\, du$

$$= \pi a^3 \int_{-1}^{1} \{1 + 4u + 4u^2 - 2u^3 - 5u^4 - 2u^5\} du$$

$$= \pi a^3 \left[2\left\{1 + \frac{4}{3} - 1\right\}\right] = \pi a^3 \cdot \frac{8}{3} = \frac{8}{3}\pi a^3 \ \Omega$$

Required surface $= \int 2\pi y \, ds$

$(ds)^2 = (dr)^2 + (r\,d\theta)^2$

$\Rightarrow \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$

$\because r = a(1+\cos\theta) \Rightarrow \frac{dr}{d\theta} = -a\sin\theta$

$\left(\frac{ds}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$

$= a^2\left(4\cos^2\frac{\theta}{2}\right) = 4a^2\cos^2\frac{\theta}{2}$

$\Rightarrow ds/d\theta = 2a\cos\theta/2$

$\rightarrow \ = 2\pi \int_{0}^{\pi} (r\sin\theta)\left(2a\cos\frac{\theta}{2}\right) d\theta$

$= 16\pi a^2 \int_{0}^{\pi} \cos^4\frac{\theta}{2} \sin\frac{\theta}{2} \, d\theta$ — (1)

Put $\cos\frac{\theta}{2} = u \Rightarrow du = -\frac{1}{2}\sin\frac{\theta}{2} d\theta$

$\left.\begin{array}{l}\theta = 0 \Rightarrow u = 1 \\ \theta = \pi \Rightarrow u = 0\end{array}\right\}$

(1) becomes $= 32\pi a^2 \int_{0}^{1} u^4 du$

$= \frac{32}{5}\pi a^2 \ \Omega$