

PHYSICS B.Sc Part-I (H) & Sub Paper-I Gr-B

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LANGRANGIAN EQUATION OF MOTION FROM D'ALEMBERT PRINCIPLE

The D'Alembert's Principle state that the total work done for the small displacement δs_i for system of particle is equal to zero

$$\sum (F_i - \dot{p}_i) = 0 \quad \text{--- (A)}$$

where F_i = applied force
 \dot{p}_i = reactional force.

Let us consider the system of particles having generalised co-ordinates $q_1, q_2, q_3, \dots, q_n$ which are independent upon each other. The position of i th particle is given by

$$s_i = s_i(q_1, q_2, q_3, \dots, q_n)$$

using Partial differentiation

$$\delta s_i = \frac{\partial s_i}{\partial q_1} \delta q_1 + \frac{\partial s_i}{\partial q_2} \delta q_2 + \frac{\partial s_i}{\partial q_3} \delta q_3 + \dots$$

$$= \sum \frac{\partial s_i}{\partial q_m} \delta q_m$$

The velocity of the particle is given by

$$V_i = \frac{\partial s_i}{\partial t} = \sum \frac{\partial s_i}{\partial q_m} \frac{\partial q_m}{\partial t} = \sum \frac{\partial s_i}{\partial q_m} \cdot \dot{q}_m$$

$$\therefore \boxed{\frac{\partial V_i}{\partial \dot{q}_m} = \frac{\partial s_i}{\partial q_m}} \quad \text{--- (B)}$$

From D'Alembert's principle.

$$\sum (F_i - \dot{p}_i) \frac{\partial s_i}{\partial q_m} \delta q_m = 0 \quad \text{--- (C)}$$

The second term of the eqnⁿ:-

$$\dot{p}_i \frac{\partial s_i}{\partial q_m} = m \ddot{s}_i \frac{\partial s_i}{\partial q_m}$$

$$= \frac{d}{dt} \left[m \dot{s}_i \frac{\partial s_i}{\partial q_m} \right] - m \dot{s}_i \frac{d}{dt} \left(\frac{\partial s_i}{\partial q_m} \right)$$

$$= \frac{d}{dt} \left[m V_i \frac{\partial V_i}{\partial \dot{q}_m} \right] - m V_i \frac{\partial}{\partial q_m} \left(\frac{d s_i}{dt} \right)$$

$$\text{or, } \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_m} = \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_m} \left(\frac{1}{2} m v_2^2 \right) \right] - m v_2 \frac{\partial v_2}{\partial \dot{q}_m}$$

$$\text{or } \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_m} = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_m} \right] - \frac{\partial}{\partial \dot{q}_m} \left[\frac{1}{2} m v_2^2 \right]$$

$$\text{or } \boxed{\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_m} = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_m} \right] - \frac{\partial T}{\partial \dot{q}_m}} \quad \text{--- (D)}$$

First term of eqnⁿ (D)

$$F_2 \frac{\partial \mathcal{L}}{\partial \dot{q}_m} = - \frac{\partial V}{\partial \dot{q}_m} \times \frac{\partial \mathcal{L}}{\partial \dot{q}_m} = - \frac{\partial V}{\partial \dot{q}_m} \quad \text{--- (E)}$$

Pulling the value of (D) & (E) in eqnⁿ (C) we get

$$\sum_n \left[- \frac{\partial V}{\partial \dot{q}_m} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_m} \right) + \left(\frac{\partial T}{\partial \dot{q}_m} \right) \right] \delta q_m = 0$$

$$\text{or, } \sum_n \left[\left(\frac{\partial (T-V)}{\partial \dot{q}_m} \right) - \frac{d}{dt} \left[\frac{\partial (T-V)}{\partial \dot{q}_m} \right] \right] \delta q_m = 0$$

∵ V does not depend upon velocity $\frac{\partial V}{\partial \dot{q}_m} = 0$

$$\therefore \sum_n \left[\frac{\partial L}{\partial \dot{q}_m} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_m} \right) \right] \delta q_m = 0$$

This equation is valid for all generalized co-ordinates

$$\boxed{\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0}$$

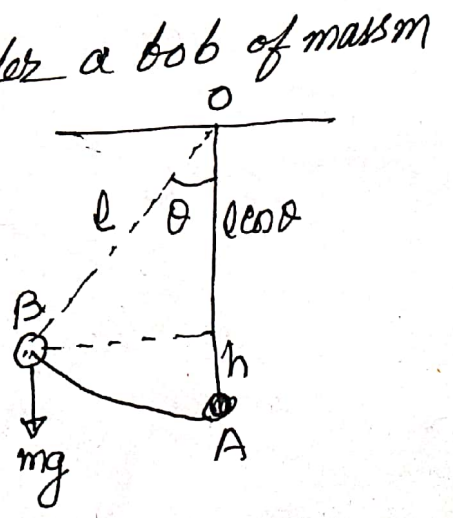
This is the Lagrangian equation of motion

Application of LEM

(1) Simple Pendulum: - Let us consider a bob of mass m is suspended from inextensible and torsion-less string of length l . It is displaced by an angle θ & left to itself.

The K.E of the bob $T = \frac{1}{2} m v^2$
 $= \frac{1}{2} m l^2 \dot{\theta}^2$

$$\boxed{T = \frac{1}{2} m l^2 \dot{\theta}^2}$$



3) The potential energy of the bob
 $V = mgh = mgl(1 - \cos \theta)$

Also Lagrangian function: $L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos \theta)$

$$\therefore \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \quad \& \quad \frac{\partial L}{\partial \theta} = mgl \sin \theta$$

Using L.E.M.

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{\theta}} \right] - \frac{\partial L}{\partial \theta} = 0$$

$$\text{or, } \frac{d}{dt} (m l^2 \dot{\theta}) + mgl \sin \theta = 0$$

$$\Rightarrow m l^2 \ddot{\theta} + mgl \sin \theta = 0$$

$$\text{or, } \ddot{\theta} = - \frac{g \sin \theta}{l}$$

For small angle θ , $\sin \theta = \theta$.

$$\ddot{\theta} = -g \theta / l$$

Hence accelⁿ is proportional to the displacement
 The motion will be simple harmonic, the time
 period of simple harmonic motion is given by

$$T = 2\pi \sqrt{\frac{\text{displ.}}{\text{accel}^n}} = 2\pi \sqrt{\theta / \ddot{\theta}}$$

$$T = 2\pi \sqrt{l/g}$$

2) Simple harmonic oscillate.

Let us consider a spring constant K .
 is connected with the mass m and the mass
 is displaced by distance x and left to it self.

$$\text{The K.E of mass } m = \frac{1}{2} m \dot{x}^2$$

$$\text{The P.E of mass } m = \frac{1}{2} K x^2$$

\therefore Lagrangian function

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$$

$$\therefore \frac{\partial L}{\partial \dot{x}} = m \dot{x} \quad \& \quad \frac{\partial L}{\partial x} = -Kx$$

Using L.E.M.

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{x}} \right] - \frac{\partial L}{\partial x} = 0$$

$$\therefore \frac{d}{dt} [m\dot{x}] + Kx = 0 \quad (4)$$

$$\text{or, } m\ddot{x} = -Kx$$

$$\text{or, } \boxed{\ddot{x} = -K \cdot \frac{x}{m}}$$

Hence the time period of S.H.M.

$$T = 2\pi \sqrt{\frac{\text{displ}}{\text{accl}}} = 2\pi \sqrt{\frac{x}{\ddot{x}}}$$

$$\boxed{T = 2\pi \sqrt{m/K}}$$

(3) Compound pendulum:- The rigid body is oscillated about a horizontal axis in vertical plane. K/A compound pendulum

$$\text{K.E, } T = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2$$

$$\text{P.E } V = -mgl \cos \theta$$

$$\text{Lagrangian function } L = T - V,$$

$$\text{or, } L = \frac{1}{2} I \dot{\theta}^2 + mgl \cos \theta$$

$$\text{Now, } \frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta} \quad \& \quad \frac{\partial L}{\partial \theta} = -mgl \sin \theta$$

\(\therefore\) the L.E.M gives us.

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{\theta}} \right] - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt} [I \dot{\theta}] + mgl \sin \theta = 0$$

For small angle \(\theta\), \(\sin \theta = \theta\).

$$\therefore I \ddot{\theta} = -mgl \theta$$

$$\text{or, } \boxed{\ddot{\theta} = -\frac{mgl}{I} \theta}$$

The time period of S.H.M.

$$T = 2\pi \sqrt{\frac{\theta}{\ddot{\theta}}} = 2\pi \sqrt{\frac{I}{mgl}}$$

$$\text{Now } I = I_c + ml^2 = mk^2 + ml^2$$

$$\therefore T = 2\pi \sqrt{\frac{mk^2 + l^2 m}{mgl}}$$

$$T = 2\pi \sqrt{\frac{k^2 + L}{g}}$$

$$\boxed{T = 2\pi \sqrt{L/g}}$$

where $L = \frac{k^2}{g} + l =$ Equivalent length of simple pendulum.

