Date:–
27/05/2020

Time:– 10a.m. to 12p.m.

Chapter:–
Hydrostatic

Topic:–
Problems of Centre Of Pressure

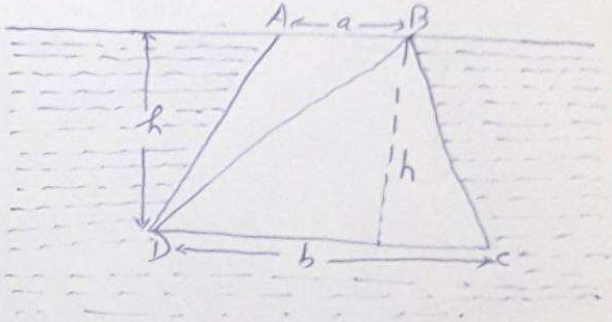
By
Professor
(Dr.) Mohammad Eqbalu zafar

Scanned by TapScanner

Ex-1 Prove that the depth of the centre of pressure of a trapezium in a liquid with side a in the surface and the parallel side b at a depth h below the surface is $\frac{a+3b}{a+2b}\cdot\frac{h}{2}$

Sol:- Join the diagonal BD

Let ρ be the density of the liquid.



The thrust on $\Delta ABD = \rho g \cdot \frac{h}{3} \cdot \frac{1}{2} ah$

$= T_1$ (say)

and the depth of the C.P. of $\Delta ABD = \frac{h}{2} = Z_1$ (say).

Also the thrust on $\Delta BCD = \rho g \cdot \frac{2h}{3} \cdot \frac{1}{2} bh = T_2$ (say).

and the depth of the C.P of $\Delta BCD = \frac{3}{4}h = Z_2$ (say).

Trapezium $ABCD = \Delta ABD + \Delta BCD$

If $\bar{z}$ be the required depth of the C.P of the trapezium ABCD.

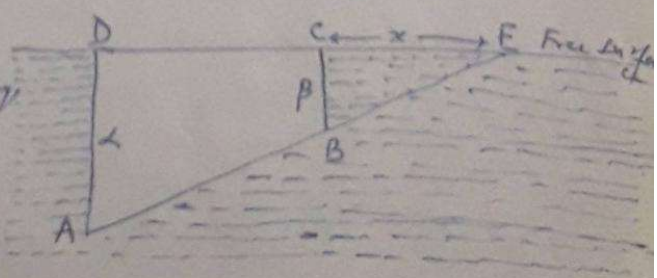
$$\bar{z} = \frac{T_1 z_1 + T_2 z_2}{T_1 + T_2}, \text{ formula}$$

$$= \frac{\frac{1}{6}\rho g h^2 a \cdot \frac{h}{2} + \frac{1}{3}\rho g h^2 b \cdot \frac{3}{4}h}{\frac{1}{6}\rho g h^2 a + \frac{1}{3}\rho g h^2 b} = \frac{\frac{1}{12}\rho g h^3 (a+3b)}{\frac{1}{6}\rho g h^2 (a+2b)}$$

$$= \frac{a+3b}{a+2b}\cdot\frac{h}{2} \quad \text{Ans}$$

Ex-2 A lamina in the shape of a quadrilateral ABCD has the side CD in the surface, and the sides AD, BC vertical and of lengths α and β respectively, Prove that the depth of its centre of pressure is $\frac{1}{2}\cdot\frac{(\alpha+\beta)(\alpha^2+\beta^2)}{\alpha^2+\alpha\beta+\beta^2}$.

Sol: Let the side AB be produced to meet the free surface in E, & let $CE = x$ (say).



Since the triangles ADE and BCE are similar, therefore $\frac{AD}{BC} = \frac{DE}{CE}$

$\Rightarrow \frac{\alpha}{\beta} = \frac{DE}{x} \quad \therefore DE = \frac{\alpha}{\beta}x$

Scanned by TapScanner

Let ρ be the density of the liquid.

Now the thrust on the $\Delta ADE = \rho g \cdot \frac{\alpha}{3} \cdot \frac{1}{2} \cdot DE \cdot AD$

$= \rho g \cdot \frac{\alpha}{3} \cdot \frac{1}{2} \cdot \frac{\alpha}{\beta} \cdot x \cdot \alpha = T_1$ (say)

and the depth of the C.P of the ΔADE

$= \frac{\alpha}{2} = z_1$ (say).

Also the thrust on the $\Delta BCE = \rho g \cdot \frac{\beta}{3} \cdot \frac{1}{2} \cdot CE \cdot BC$

$= \rho g \cdot \frac{\beta}{3} \cdot \frac{1}{2} x \beta = T_2$ (say).

and the depth of the C.P. of the $\Delta BCE = \frac{\beta}{2} = z_2$ (say).

Quadrilateral $ABCD = \Delta AED - \Delta BEC$.

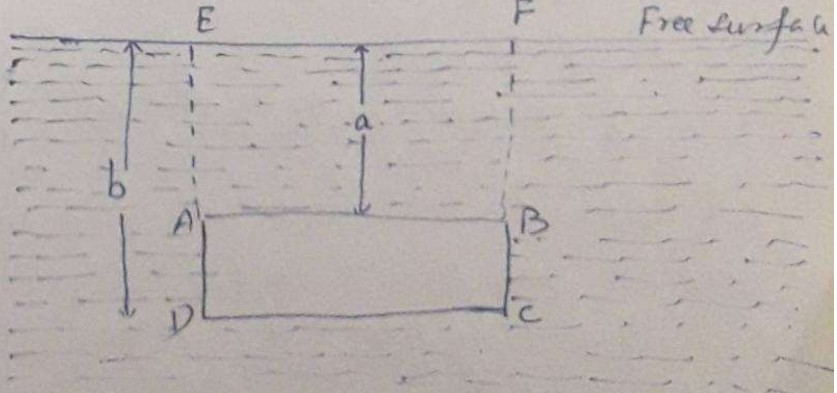
If $\bar{z}$ be the required depth of the C.P of the quadrilateral ABCD, $\bar{z} = \frac{T_1 z_1 - T_2 z_2}{T_1 - T_2}$, formula.

$$= \frac{\frac{1}{6}\rho g \frac{\alpha^3 x}{\beta} \cdot \frac{\alpha}{2} - \frac{1}{6}\rho g \beta^2 x \cdot \frac{\beta}{2}}{\frac{1}{6}\rho g \frac{\alpha^3 x}{\beta} - \frac{1}{6}\rho g \beta^2 x}$$

$$= \frac{\frac{1}{12}\rho g x \left(\frac{\alpha^4}{\beta} - \beta^3\right)}{\frac{1}{6}\rho g x \left(\frac{\alpha^3}{\beta} - \beta^2\right)} = \frac{1}{2} \cdot \frac{\alpha^4 - \beta^4}{\alpha^3 - \beta^3}$$

$$= \frac{1}{2} \frac{(\alpha-\beta)(\alpha+\beta)(\alpha^2+\beta^2)}{(\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)} = \frac{1}{2} \frac{(\alpha+\beta)(\alpha^2+\beta^2)}{\alpha^2+\alpha\beta+\beta^2}$$

Ex-3 Show that the depth below the surface of a liquid of the centre of pressure of a rectangle two of whose sides are horizontal and at depths a and b is $\frac{2}{3} \cdot \frac{a^2+ab+b^2}{a+b}$



Soln:- Produce DA and CB to meet the free surface in E and F respectively

Let $AB = h = CD$, and

ρ = the density of the liquid

Now the rectangle ABCD = rectangle EFCD − rectangle EFBA.

Scanned by TapScanner

The thrust on the rectangle $EFCD = \rho g \cdot \frac{b}{2} \cdot hb = T_1$ (say),
and the depth of the C.P. $= \frac{2}{3}b = z_1$ (say).

The thrust on the rectangle $EFBA = \rho g \cdot \frac{a}{2} \cdot ha = T_2$ (say),
and the depth of the C.P $= \frac{2}{3}a = z_2$ (say).

If $\bar{z}$ be the depth of the centre of pressure of the rectangle ABCD, $\bar{z} = \frac{T_1 z_1 - T_2 z_2}{T_1 - T_2}$, (formula)

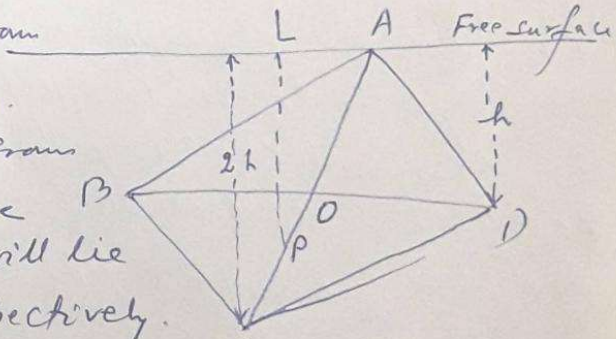
i.e. $$\bar{z} = \frac{\frac{1}{2}\rho g b^2 h \cdot \frac{2}{3}b - \frac{1}{2}\rho g a^2 h \cdot \frac{2}{3}a}{\frac{1}{2}\rho g b^2 h - \frac{1}{2}\rho g a^2 h}$$

or $$\bar{z} = \frac{2}{3} \cdot \frac{\rho g h (b^3 - a^3)}{\rho g h (b^2 - a^2)} = \frac{2}{3} \frac{(b-a)(b^2+ba+a^2)}{(b-a)(b+a)}$$

Hence $$\bar{z} = \frac{2}{3} \cdot \frac{b^2+ba+a^2}{b+a}$$

Ex-4 A parallelogram ABCD is immersed in a liquid with A in the surface and BD horizontal. Prove that the centre of Pressure (C.P) P lies on A such that AP : AC = 7 : 12.

Sol:- Here the area of the parallelogram
= area of $\triangle ABD$ + area of $\triangle BCD$.



Since the diagonals of a parallelogram bisect each other, therefore the centre of pressure of $\triangle^s ABD$ and BCD will lie on the medians AO and CO respectively.
Hence the C.P of the parallelogram will also lie on AC.

$\because \triangle ABD \equiv \triangle BCD$,

$\therefore$ area of $\triangle ABD$ = area of $\triangle BCD = S$ (say).

Let ρ be the density of the liquid, and h be the depth of BD below the free surface.
Then 2h is the depth of C below the free surface.
Now the thrust on $\triangle ABD = \rho g \cdot \frac{2}{3}h \cdot S = T_1$ (say)
and the depth of the C.P of $\triangle ABD = \frac{3}{4}h = z_1$ (say).
Also the thrust on $\triangle BCD = \rho g \left(\frac{h+h+2h}{3}\right) S = \frac{4}{3}\rho g h \cdot S = T_2$ (say)

and the depth of the C.P of $\Delta BCD = \frac{\alpha^2+\beta^2+\gamma^2+\alpha\beta+\beta\gamma+\gamma\alpha}{2(\alpha+\beta+\gamma)}$ (formula)

$= \frac{h^2+h^2+(2h)^2+h\cdot h+h\cdot 2h+2h\cdot h}{2(h+h+2h)} = \frac{11}{8}h = z_2$ (say).

Hence the depth of P, the C.P. of the parallelogram ABCD below the surface. $= \frac{T_1 z_1 + T_2 z_2}{T_1+T_2}$ (formula)

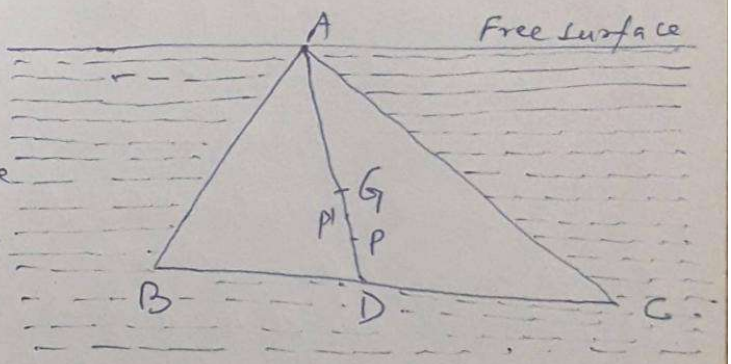
i.e. $PL = \frac{\frac{2}{3}\rho g h\cdot s\cdot\frac{3}{4}h + \frac{4}{3}\rho g h\cdot s\cdot\frac{11}{8}h}{\frac{2}{3}\rho g h\cdot s + \frac{4}{3}\rho g h\cdot s} = \frac{\rho g h^2\cdot s\left(\frac{1}{2}+\frac{11}{6}\right)}{\rho g h\cdot s\left(\frac{2}{3}+\frac{4}{3}\right)} = \frac{7}{6}h$

Also $CM = 2h$.

$\because \Delta^s APL$ and ACM are similar.

$\therefore \frac{AP}{AC} = \frac{PL}{CM} = \frac{\frac{7}{6}h}{2h} = \frac{7}{12}$ Ans

Ex 5 A triangle of height h is immersed in a liquid with the base horizontal and vertex in the surface. If the atmospheric pressure is equivalent to a head of H feet of the liquid, prove that the centre of pressure is raised to a height $\frac{hH}{4(2h+3H)}$ in the plane of the triangle.



Sol:- If the atmospheric pressure be neglected, then the depths of the C.G and the C.P of the triangle ABC below the free surface are $\frac{2}{3}h$ and $\frac{3}{4}h$ respectively, where h is the depth of BC.

If the atmospheric pressure is taken into consideration, then the thrusts acting on the triangle are:

(i) the thrust $\rho g\cdot\frac{2}{3}h\cdot\Delta$ acting at P, where Δ = area of ΔABC, and depth of P below $A = \frac{3}{4}h$, and

an additional thrust $\rho g H\cdot\Delta$ acting at G, where H is the height of the same liquid superimposed on the surface of the given liquid, and depth of G below $A = \frac{2}{3}h$.

If $\bar{z}$ be the depth of C.P (P') in the new position, $\bar{z} = \frac{\rho g\cdot\frac{2}{3}h\cdot\Delta\,\frac{3}{4}h + \rho g H\cdot\frac{2}{3}h}{\rho g\cdot\frac{2}{3}h\Delta + \rho g H\cdot\Delta}$

$= \frac{\rho g h\Delta\left(\frac{h}{2}+\frac{2H}{3}\right)}{\rho g\Delta\left(\frac{2h}{3}+H\right)}$

$= \frac{h}{2}\cdot\frac{3h+4H}{2h+3H}$

Hence the height through which the C.P is raised = depth of P − depth of P'

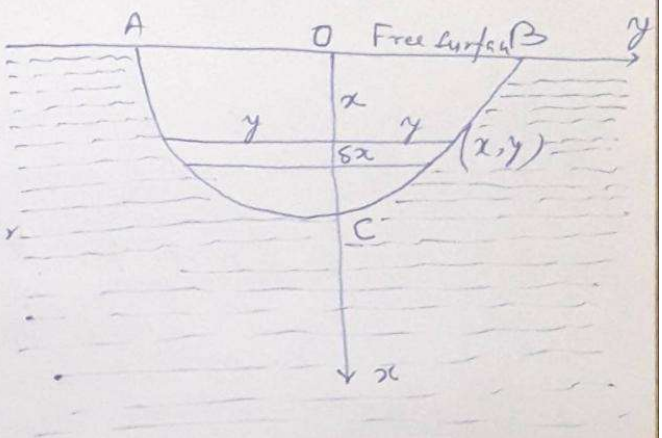
$= \frac{3}{4}h - \frac{h}{2}\cdot\frac{3h+4H}{2h+3H}$

$= \frac{hH}{4(2h+3H)}$ Ans

Ex-6 A semi-circular lamina is immersed in a liquid with the diameter in the surface. Find the centre of pressure.

Soln:- Let OC be a radius of the circle perpendicular to AOB.

Take OC and OB and x and y axes respectively.



Then the equation of the circular arc is $x^2 + y^2 = a^2$ ——— (1)

where a is the radius of the circle.

The lamina is symmetrical about OC, therefore, by symmetry, the C.P. of the lamina will lie on OC. $\therefore \bar{y} = 0$

The pressure on the elementary strip is at the middle point whose co-ordinates are $\left(x, \frac{y}{2}\right)$.

$$\therefore \bar{y} = \frac{\int_0^a \frac{y}{2} \cdot \rho g x \cdot \sqrt{a^2 - x^2}\, dx}{\int_0^a \rho g x \cdot \sqrt{a^2 - x^2}\, dx} = \frac{1}{2} \frac{\int_0^a \sqrt{a^2 - x^2} \cdot x \cdot \sqrt{a^2 - x^2}\, dx}{\int_0^a x \sqrt{a^2 - x^2}\, dx}$$

$$= \frac{1}{2} \cdot \frac{\int_0^{\pi/2} a \sin\theta (a^2 - a^2 \sin^2\theta) \cdot a \cos\theta\, d\theta}{\int_0^{\pi/2} a \sin\theta \sqrt{a^2 - a^2 \sin^2\theta} \cdot a \cos\theta\, d\theta}$$

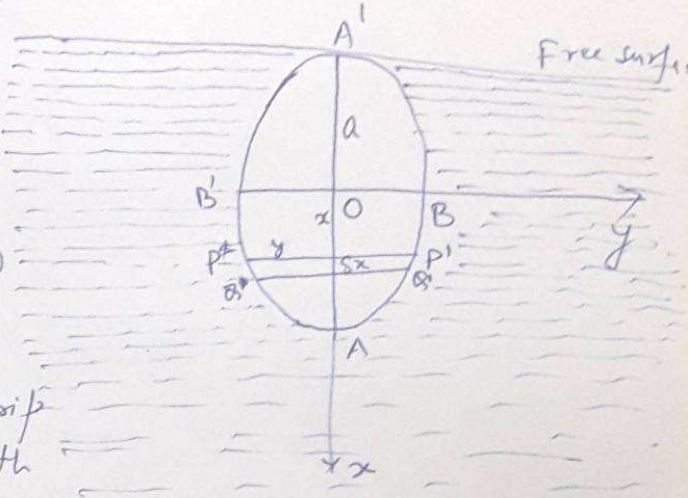
$$= \frac{a}{2} \cdot \frac{\int_0^{\pi/2} \sin\theta \cos^3\theta\, d\theta}{\int_0^{\pi/2} \sin\theta \cos^2\theta\, d\theta} = \frac{a}{2} \cdot \frac{\left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/2}}{\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2}} = \frac{3}{8} a$$

Hence the co-ordinates of C.P of the quadrant are $\left(\frac{3\pi a}{16}, \frac{3a}{8}\right)$.

Ex-7 An ellipse is just immersed with its major axis vertical. Show that if C.P coincides with the lower focus, the eccentricity of the ellipse is $1/4$.

Sol: Taking the major and minor axes of the ellipse as the axes of co-ordinates, the eqn of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2} \quad \text{---} (1)$$



where $AA' = 2a$ and

$BB' = 2b$.

Now consider an elementary strip $PP'Q'Q$ of thickness δx at a depth x below O, the centre of the ellipse.

Then δS = area of the strip

$$= 2y\,\delta x = \frac{2b}{a}\sqrt{a^2 - x^2}\,\delta x, \quad [\text{from } (1)]$$

and p = pressure per unit area $= \rho g(a + x)$

Clearly the C.P of the ellipse lies on the major axis, $\therefore \bar{y} = 0$.

If $\bar{x}$ be the depth of C.P. of the ellipse from O, then

$$\bar{x} = \frac{\int x p\, ds}{\int p\, ds} = \frac{\int_{-a}^{a} x \rho g(a+x)\frac{2b}{a}\sqrt{a^2 - x^2}\,dx}{\int_{-a}^{a} \rho g(a+x)\frac{2b}{a}\sqrt{a^2 - x^2}\,dx}$$

$$= \frac{a\int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx + \int_{-a}^{a} x^2\sqrt{a^2 - x^2}\,dx}{a\int_{-a}^{a}\sqrt{a^2 - x^2}\,dx + \int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx}$$

If $f(x)$ is an odd function of x, then $\int_{-a}^{a} f(x)\,dx = 0 \Rightarrow \int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx = 0$

$$\therefore \bar{x} = \frac{\int_{-a}^{a} x^2\sqrt{a^2 - x^2}\,dx}{a\int_{-a}^{a}\sqrt{a^2 - x^2}\,dx} = \frac{a\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta}{\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta}$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$
with $x = a, \theta = \pi/2$
$x = -a, \theta = -\pi/2$

$$= \frac{a}{4}\cdot\frac{\int_{-\pi/2}^{\pi/2}\frac{1-\cos 4\theta}{2}\,d\theta}{\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta} = \frac{a}{4}\cdot\frac{\pi}{\pi} = \frac{a}{4}$$

But the C.P. coincides with the focus. $\therefore \bar{x} = ae$. i.e. $\frac{a}{4} = ae$

Hence the eccentricity of the ellipse $= 1/4$

Scanned by TapScanner