PHYSICS
B.Sc Part-III (H) Paper-V Gr-B
By :- Dr. Mirza Abbas Hussain

CLASSICAL MECHANICS

Hamilton's Principle, Lagranges equatics of motions

Hamilton's Canonical Equation

CLASSICAL MECHNICS.

Hamilton's Principle :-

The Principle states that for a conservative system, its motion from time t_1 to time t_2 is such that the line integral of Lagrangian 'L' is extrimum value for any path of the motion

$$I = \int_{t_1}^{t_2} L\,dt = \text{extrimum}.$$

The Principle may be expressed as

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0 \qquad (A)$$

Lagranges equation of motion from Hamilton's Principle.

The Lagranges function L is a function of Generalised co-ordinate q_i, Generalised velocity ($\dot{q}_i$) and time 't'

$$L = L(q_i, \dot{q}_i, t).$$

If the Lagrangian does not depends upon time, the variation δL is given by

$$\delta L = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i$$

Integrating both side we get

$$\int_{t_1}^{t_2} \delta L\,dt = \int_{t_1}^{t_2} \sum_i \frac{\partial L}{\partial q_i}\delta q_i\,dt + \int_{t_1}^{t_2} \sum \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\,dt$$

$$\text{or,} \quad 0 = \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i}\delta q_i\,dt + \int_{t_1}^{t_2} \sum \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\,dt.$$

or, $$\int_{t_1}^{t_2}\sum_{i}^{n}\frac{\partial L}{\partial q_i}\delta q_i\,dt+\left|\sum_{i}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta q_i\,t\right|_{t_1}^{t_2}-\int_{t_1}^{t_2}\sum_{i}^{n}\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right]\delta q_i\,dt$$

$$=0 \quad \text{—} \quad (B)$$

At the end points of both $\delta q_i(t_1) = \delta q_i\, t_2 = 0$

$$\therefore \left|\sum_{i}\frac{\partial L}{\partial \dot{q}_i}\delta q_i\,t\right|_{t_1}^{t_2} = 0$$

Therefore eqn (B) takes the form:

$$\int_{t_1}^{t_2}\sum_{i}\frac{\partial L}{\partial q_i}\delta q_i\,dt-\int_{t_1}^{t_2}\sum_{i}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\,dt=0$$

or $$\sum_{i}\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\left(\frac{\partial L}{\partial q_i}\right)\right]\delta q_i\,dt=0 \quad (C)$$

For holonomic system, the generalised co-ordinates δq_i are independent of each other. Therefore the coefficient of each δq_i must vanishes.

$$\boxed{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right]-\left[\frac{\partial L}{\partial q_i}\right]=0} \quad \text{———} \quad (D)$$

This is Lagranges equation of motion.

Application

(1) compound pendulum :- Let us consider a compound pendulum, at an angle 'θ' from the vertical axis -

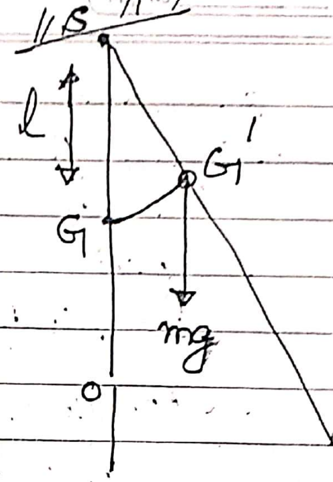
The K.E of oscillating system, $T = \frac{1}{2} I \dot{\theta}^2$

Potential energy with respect to plane S is

$$V = -mgl\cos\theta$$

Lagrangian; $L = T - V$

or, $L = \frac{1}{2} I \dot{\theta}^2 + mgl\cos\theta$



Now, $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$

& $\frac{\partial L}{\partial \dot{\theta}} = I\ddot{\theta}$

The Lagrangian equation in coordinate θ.

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = 0$$

$$I\ddot{\theta} + mgl\theta$$

$$\boxed{\ddot{\theta} = -\frac{mgl}{I}\theta}$$

This is the equ^n of S.H.M. The Time period of S.H.M.

$$T = 2\pi\sqrt{\theta/\ddot{\theta}} = 2\pi\sqrt{\frac{I}{mgl}}$$

or, $T = 2\pi\sqrt{\frac{I_c + ml^2}{mgl}}$

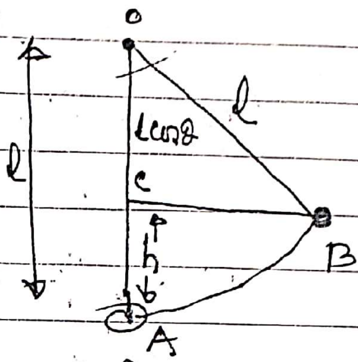
$$= 2\pi\sqrt{\frac{mK^2 + ml^2}{mgl}}$$

or $T' = 2\pi\sqrt{\frac{K^2/l + l}{g}}$

$$\boxed{T = 2\pi\sqrt{L/g}}$$

2) Simple pendulum:- Let us consider a simple pendulum of length l and mass m of the bob at an angle θ from vertical.
The K.E of the bob.

$$T = \frac{1}{2} m\dot{x}^2 = \frac{1}{2} m(l\dot{\theta})^2$$



The Potential energy of the bob.

$$V = mgh = mgl(1-\cos\theta)$$

Lagrangian function,

$$L = T - V$$

$$L = \frac{1}{2} m\dot{\theta}^2 l^2 - mgl(1-\cos\theta)$$

$$\therefore \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad \& \quad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

Lagrangian equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0$$

or, $$\frac{d}{dt}(m\dot{\theta}l^2) + mgl\sin\theta = 0$$

or, $$l^2\ddot{\theta} = -gl\theta$$

or, $$\boxed{\ddot{\theta} = -\frac{g}{l}\theta}$$

The time period of S.H.M.

$$T = 2\pi\sqrt{\theta/\ddot{\theta}}$$

or, $$\boxed{T = 2\pi\sqrt{l/g}}$$

State Hamilton's Principle and derive Lagrange's equation of motion. Give its some application

HAMILTON'S PRINCIPLE AND HAMILTON CANONICAL EQUATION

Hamilton Principle:- this Principle state that for a conservative system, the motion of particle from time t_1 to time t_2 is such that line integral of Lagrangian (L) is extremum value for any path of motion

$$I = \int_{t_1}^{t_2} L\,dt = \text{Extremum.}$$

Where $L = T - V$ = Lagrangian function

It may be stated as.

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0$$

Explanation:- It is to be noted that for conservative system, the position and velocity are function of time and the kinetic energy is a function of velocity & position.

$$L = L(q_1, q_2 \dots\dots, \dot{q}_1, \dot{q}_2 \dots\dots, t).$$

$$\therefore \delta I = \delta \int_{t_1}^{t} L(q_i, \dot{q}_i, t)\,dt = 0.$$

Using δ-variation, the Euler Lagranges equation is given by

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0.$$

Where $i = 1, 2, 3 \dots\dots$

In terms of Lagrangian Hamilton's Principle state that "of all the possible path along which a dynamical system may move from one point to another point in the configuration space within a given interval of time, the actual path followed is that for which the time integral of the Lagrangian function for the system is an extremum.

Hamilton's (canonical) equation of motion

Hamilton's Principle states that

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0 \quad \text{——— (1)}$$

The Hamilton's function is given by

$$H = \sum_i p_i \dot{q}_i - L$$

or, $L = \sum_i p_i \dot{q}_i - H(q_i, p_i, t)$

From equal (1)

$$\delta \int_{t_1}^{t_2} \left[\sum p_i \dot{q}_i - H(q_i, p_i, t)\right] dt = 0$$

or, $\delta \sum_i \int_{t_1}^{t_2} p_i \frac{\partial q_i}{\partial t} dt - \delta \int_{t_1}^{t_2} H \cdot dt = 0$

or, $\delta \sum_i \int_{t_1}^{t_2} p_i \, \partial q_i - \delta \int_{t_1}^{t_2} H dt = 0$ ——— (2)

For all possible path in configuration space with parameter α, the δ variation is

$$\delta \to d\alpha \cdot \frac{\partial}{\partial \alpha} \quad \text{------} \quad (3)$$

$$\therefore \delta I = d\alpha \frac{\partial I}{\partial \alpha} = d\alpha \frac{d}{d\alpha} \int_{t_1}^{t_2} \sum p_i \dot{q}_i - H(p_i, q_i, t)\, dt = 0$$

$$\text{or, } d\alpha \int_{t_1}^{t_2} \Big[\sum p_i \cdot \frac{\partial \dot{q}_i}{\partial \alpha} + \frac{\partial p_i}{\partial \alpha} \dot{q}_i - \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial \alpha} - \frac{\partial H}{\partial q_i} \cdot \frac{\partial q_i}{\partial \alpha} - \frac{\partial H}{\partial t} \cdot \frac{\partial t}{\partial \alpha} \Big] dt = 0 \quad (4)$$

Again $\frac{\partial t}{\partial \alpha} = 0$, since time of interval along every path is same.

$$\text{Also } \int_{t_1}^{t_2} p_i \cdot \frac{\partial \dot{q}_i}{\partial \alpha} = \int_{t_1}^{t_2} p_i \frac{d}{dt} \cdot \left(\frac{\partial q_i}{\partial \alpha} \right) dt$$

$$= p_i \frac{\partial q_i}{\partial \alpha} - \int_{t_1}^{t_2} \dot{p}_i \frac{\partial q_i}{\partial \alpha}\, dt$$

$$= 0 - \int_{t_1}^{t_2} \dot{p}_i \frac{\partial q_i}{\partial \alpha}\, dt \quad \text{---} \quad (5)$$

From eqn (4)

$$\delta I = d\alpha \int_{t_1}^{t_2} \sum_i \left[\frac{\partial p_i}{\partial \alpha} \dot{q}_i - \dot{p}_i \frac{\partial q_i}{\partial \alpha} - \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial \alpha} - \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial \alpha} \right] dt \quad \text{---} \quad (6)$$

Putting $\frac{\partial q_i}{\partial \alpha} d\alpha = \delta q_i$

& $\frac{\partial p_i}{\partial \alpha} d\alpha = \delta p_i$

We get

$$0 = \sum \int_{t_1}^{t_2} \left[\delta p_i \dot{q}_i - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right] dt$$

or $\sum_{i} \int_{t_1}^{t_2} \left[\delta p_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) - \delta q_i \left(\dot{p}_i - \frac{\partial H}{\partial q_i} \right) \right] dt = 0$ — (7)

Since p_i and q_i are independent Variable their Variation δp_i & δq_i will also be independent of each other. So above integral can be Vanish only if the coeffecient seperately Vanishes.

$$\therefore \boxed{\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

This is required canonical equation

State and explain Hamilton (Variational) Principle. obtain Hamilton canonical equation.