LANGRANGIAN EQUATION OF MOTION FROM D'ALEMBERT PRINCIPLE

The D'Alemberts Principle state that the total workdone for the small displacement δr_i for system of particle is equal to zero

$$\sum(F_i - \dot{P}_i) = 0 \quad \text{———— (A)}$$

where F_i = applied force
$\dot{p}_i$ = reactional force.

Let us consider the system of particles having generalised co-ordinates $q_1, q_2, q_3 \text{------} q_n$ which are independent upon each other. The position of ith particle is given by

$$r_i = r_i(q_1, q_2, q_3 \text{-----} q_m)$$

using Partial differentiation

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\delta q_1 + \frac{\partial r_i}{\partial q_2}\partial q_2 + \frac{\partial r_i}{\partial q_3}\delta q_3 + \cdots$$

$$= \sum \frac{\partial r_i}{\partial q_m}\delta q_m.$$

The velocity of the particle is given by

$$V_i = \frac{\partial r_i}{\partial t} = \sum \frac{\partial r_i}{\partial q_m}\cdot\frac{\delta q_m}{\delta t}$$

$$= \sum \frac{\partial r_i}{\partial q_m}\cdot \dot{q}_m$$

$$\therefore \boxed{\frac{\partial V_i}{\partial \dot{q}_m} = \frac{\partial r_i}{\partial q_m}} \quad \text{———— (B)}$$

from D'Alemberts principl.

$$\sum(F_i - \dot{p}_i)\frac{\partial r_i}{\partial q_m}\delta q_m = 0 \quad \text{———— (C)}$$

The second term of the equ^n:-

$$\dot{p}_i\frac{\partial r_i}{\partial q_m} = m\ddot{r}_i\frac{\partial r_i}{\partial q_m}$$

$$= \frac{d}{dt}\left[\left(m\dot{r}_i\frac{\partial r_i}{\partial q_m}\right)\right] - m\dot{r}_i\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_m}\right)$$

$$= \frac{d}{dt}\left[mV_i\frac{\partial V_i}{\partial \dot{q}_m}\right] - mV_i\frac{\partial}{\partial q_m}\left(\frac{dr_i}{dt}\right)$$

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or, $\dot{p}_2 \frac{\partial r_2'}{\partial q_n} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_n}\left(\frac{1}{2} m V_2^2\right)\right] - m V_2 \frac{\partial V_2'}{\partial q_n}$

or $\dot{p}_2 \frac{\partial r_2'}{\partial q_n} = \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_n}\right] - \frac{\partial}{\partial q_n}\left[\frac{1}{2} m V_2^2\right]$

or, $\boxed{\dot{p}_2 \frac{\partial r_2}{q_n} = \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_n}\right] - \frac{\partial}{\partial q_n}[T]}$ —— (D)

First Term of equ^n (C)

$F_2 \frac{\partial r_2'}{\partial q_n} = -\frac{\partial V}{\partial r_2} \times \frac{\partial r_2'}{\partial q_n} = -\frac{\partial V}{\partial q_n}$ —— (E)

Putting the value of (D) & (E) in equ^n (C) we get

$$\sum_n \left[-\frac{\partial V}{\partial q_n} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) + \left(\frac{\partial T}{\partial q_n}\right)\right] \delta q_n = 0$$

or, $$\sum_n \left\{\left(\frac{\partial (T-V)}{\partial q_n}\right) - \frac{d}{dt}\left[\frac{\partial (T-V)}{\partial \dot{q}_n}\right]\right\} \delta q_n = 0$$

$\because$ V does not depends upon velocity $\frac{\partial V}{\partial \dot{q}_n} = 0$

$$\therefore \sum_n \left[\frac{\partial L}{\partial q_n} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right)\right] \delta q_n = 0$$

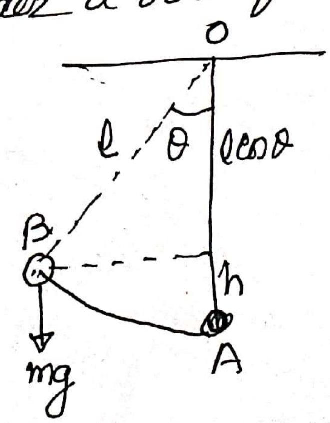
This equation is valid for all generalised co-ordinate

$$\boxed{\left(\frac{\partial L}{\partial q}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}$$

This is K/a Lagrangian equation of Motion

<u>Application of LEM</u>

(1) Simple Pendulum :- Let us consider a bob of mass m is suspended from inextensible and torsion-less string of length l. It is displaced by an angle θ & left to itself.



The K.E of the bob $T = \frac{1}{2} m v^2$

$= \frac{1}{2} m l^2 \omega^2$

$$\boxed{T = \frac{1}{2} m l^2 \dot{\theta}^2}$$

The potential energy of the bob

$$V = mgh = mgl(1-\cos\theta).$$

Now Lagrangian function:

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

$$\therefore \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \ \& \ \frac{\partial L}{\partial \theta} = mgl\sin\theta.$$

Using L.E.M.

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\text{or, } \frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\text{or, } \boxed{\ddot{\theta} = -\frac{g\sin\theta}{l}.}$$

For small angle θ, $\sin\theta = \theta$.

$$\ddot{\theta} = -g\theta/l.$$

Hence accel^n is proportional to the displacement, the motion will be simple harmonic, the time period of Simple Harmonic motion is given by

$$T = 2\pi\sqrt{\frac{\text{disp.}}{\text{accel}^n}} = 2\pi\sqrt{\theta/\ddot{\theta}}$$

$$\boxed{T = 2\pi\sqrt{l/g}.}$$

2) Simple harmonic oscillator.

Let us consider a spring constant K. is connected with the mass m and the mass is displaced by distance (x) and left to itself.

The K.E of mass m $= \frac{1}{2}m\dot{x}^2$

The P.E of mass m $= \frac{1}{2}Kx^2$.

$\therefore$ Lagrangian function

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}Kx^2$$

$$\therefore \frac{\partial L}{\partial \dot{x}} = m\dot{x} \ \& \ \frac{\partial L}{\partial x} = -Kx.$$

Using L.E.M.

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} = 0$$

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$\therefore \frac{d}{dt}[m\dot{x}] + Kx = 0$ (4)

or, $m\ddot{x} = -Kx$

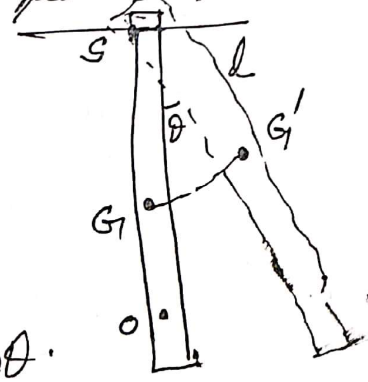
or, $\boxed{\ddot{x} = -K \cdot \frac{x}{m}}$

Hence the time period of S.H.M.

$T = 2\pi\sqrt{\frac{disp}{accl^n}} = 2\pi\sqrt{\frac{x}{\ddot{x}}}$

$$\boxed{T = 2\pi\sqrt{m/K}}$$

(3) Compound pendulum:- The rigid body is oscillated about a horizontal axis in vertical plane k/a compound pendulum.



$K.E,\ T = \frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{\theta}^2$

$P.E\ V = -mgl\cos\theta$

Lagrangian function $L = T - V$,

or, $L = \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta$.

Now, $\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$ & $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$.

$\therefore$ The L.E.M gives us.

$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = 0$

$\frac{d}{dt}[I\dot{\theta}] + mgl\sin\theta = 0$

For small angle θ, $\sin\theta = \theta$.

$\therefore I\ddot{\theta} = -mgl\,\theta$.

or, $\boxed{\ddot{\theta} = -\frac{mgl}{I}\theta}$

The time period of S.HM.

$T = 2\pi\sqrt{\theta/\ddot{\theta}} = 2\pi\sqrt{\frac{I}{mgl}}$

Now $I = I_c + ml^2 = mK^2 + ml^2$

$\therefore T = 2\pi\sqrt{\frac{mK^2 + l^2 m}{mgl}}$.

$T = 2\pi\sqrt{\frac{K^2/l + l}{g}}$

$$\boxed{T = 2\pi\sqrt{L/g}}.$$

Where $L = \frac{K^2}{l} + l$ = Equivalent length of simple pendulum.

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