By :- Dr. Mirza Abbas Hussain

ANGULAR MOMENTUM OPRATOR

The angular momentum "L" is given by:
$$\vec{L} = \vec{r} \times \vec{p}$$

where r & p represents the position vector and the linear momentum vector.
In cartesian co-ordinate

$$\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$$
$$\& \; p = \vec{i}P_x + \vec{j}P_y + \vec{k}P_z$$

So,
$$L = \vec{r} \times \vec{p} = \begin{vmatrix} i & j & k \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix} \quad \text{—— (1)}$$

$$= \vec{i}[yP_z - zP_y] + j(zP_x - xP_z) + k(xP_y - yP_x)$$

The components of angular momentum will therefore, be given by

$$L_x = (yP_z - zP_y)$$
$$L_y = zP_x - xP_z$$
$$L_z = xP_y - yP_x$$

The opperation for P_x, P_y, P_z are $-i\hbar\frac{\partial}{\partial x}$, $-i\hbar\frac{\partial}{\partial y}$ and $i\hbar\frac{\partial}{\partial z}$ respectively.

$$\therefore \hat{L}_x = i\hbar\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]$$
$$\hat{L}_y = i\hbar\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right] \quad \text{—— (2)}$$
$$\text{and } \hat{L}_z = i\hbar\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]$$

Commutation Rules for angular momentum

[A] Commutation rules for the angular momentum operators can be easily obtained by using the commutation relation between co-ordinate and momentum operators

$$[\hat{x}, \hat{y}] = [\hat{y}, \hat{z}] = [\hat{z}\, \hat{x}] = 0$$

$$[\hat{P}_x\, \hat{P}_y] = [\hat{P}_y\, \hat{P}_z] = [\hat{P}_z\, \hat{P}_x] = 0$$

$$[\hat{P}_x\, \hat{y}] = [\hat{P}_y\, z] = [\hat{P}_z\, x] = 0$$

and $[\hat{P}_x\, \hat{x}] = [\hat{P}_y\, \hat{y}] = [P_z\, \hat{z}] = -i\hbar$

The angular momentum operators are defined by

$$L_x = [\hat{y}\hat{P}_z - \hat{z}\hat{P}_y]$$

$$L_y = [\hat{z}\hat{P}_x - \hat{x}\hat{P}_z]$$

and $L_z = [\hat{x}\hat{P}_y - \hat{y}\hat{P}_x]$

Commutator of L with the co-ordinate x, y, z may be given by

$$[L_x\, \hat{y}] = [\hat{L}_x y - \hat{y}\hat{L}_x]$$

$$= (y\hat{P}_z - z\hat{P}_y)y - y(y\hat{P}_z - z\hat{P}_y)$$

$$= \hat{y}\hat{P}_z\hat{y} - \hat{z}\hat{P}_y\hat{y} - \hat{y}\hat{y}\hat{P}_z - \hat{y}\hat{z}\hat{P}_y$$

$$= yP_z y - yyP_z - zP_y y - \cancel{zy}P_y$$

$$= \hat{y}[\hat{P}_z, \hat{y}] - z[\hat{P}_y, \hat{y}]$$

$$= 0 \quad - (-i\hbar)\hat{z}$$

$$= i\hbar \hat{z}$$

Similarly $[L_y, \hat{z}] = i\hbar \hat{x}$

$[L_z \hat{x}] = i\hbar \hat{y}$

and $[L_x, x] = [L_y y] = [L_z z] = 0$ — A

[B] commulation rule of the angular momentum operator $\hat{L}_x$ & momentum operator $\hat{P}_y$

$$[L_x, \hat{P}_y] = [L_x P_y - P_y \hat{L}_x]$$

$$= (yP_z - zP_y)\hat{P}_y - \hat{P}_y(yP_z - zP_y)$$

$$= y\hat{P}_z\hat{P}_y - z\hat{P}_y\hat{P}_y - \hat{P}_y\hat{y}\hat{P}_z + \hat{P}_y\hat{z}P_y$$

$$= \hat{y}\hat{P}_y\hat{P}_z - \hat{P}_y\hat{y}\hat{P}_z + \cancel{\hat{P}_y\hat{z}\hat{P}_y} - \cancel{z\hat{P}_y\hat{P}_y}$$

$\because P_z P_y = P_y P_z$

& $\hat{z}\hat{P}_y = \hat{P}_y\hat{z}$

$$\therefore [L_x \hat{P}_y] = [\hat{y}, \hat{P}_y]\hat{P}_z = i\hbar P_z$$

Similarly $L_y P_{\cancel{z}} = i\hbar P_x$

$L_z P_x = i\hbar P_y$ — B

and $[L_x P_x] = [\hat{L}_y P_y] = [\hat{L}_z \hat{P}_z] = 0$

(c) Commutation rule between Angular momentum operators L_x L_y L_z

commutation of $\hat{L_x}$ and $\hat{L_y}$

$$[L_x, L_y] = (\hat{L_x}\hat{L_y} - \hat{L_y}\hat{L_x})$$

$$= \left[-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]\left[-i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]$$

$$- \left[-i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]\left[-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]$$

$$= -\hbar^2\left[y\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial}\left(x\frac{d}{dz}\right) - z\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x}\right) + z\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial z}\right)\right] + \hbar^2\left[x\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial z}\right) - z\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial y}\right)\right.$$

$$\left. - x\frac{\partial}{\partial z}\left(y\frac{\partial}{\partial z}\right) + x\frac{\partial}{\partial z}\, z\frac{\partial}{\partial y}\right]$$

$$= -\hbar^2\left[y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial z\partial x} - yx\frac{d^2}{dz^2} - z^2\frac{\partial^2}{\partial y\partial x} + zx\frac{\partial^2}{\partial y\partial z}\right.$$

$$\left. + \hbar^2\left[zy\frac{\partial^2}{\partial x\partial z} - z^2\frac{\partial^2}{\partial x\partial y} - xy\frac{\partial^2}{\partial z^2} + x\frac{\partial}{\partial y} + xz\frac{\partial^2}{\partial z\partial y}\right]\right.$$

$$= \hbar^2\left[y\frac{\partial}{\partial x} + \cancel{yz\frac{\partial^2}{\partial z\partial x}} - \cancel{yx\frac{d^2}{dz^2}} - \cancel{z^2\frac{\partial^2}{\partial y\partial x}} + \cancel{zx\frac{\partial^2}{\partial y\partial z}}\right.$$

$$\left. - \cancel{zy\frac{\partial^2}{\partial x\partial z}} + \cancel{z^2\frac{\partial^2}{\partial x\partial y}} + \cancel{xy\frac{\partial^2}{\partial z^2}} - x\frac{\partial}{\partial y} - \cancel{xz\frac{\partial^2}{\partial z\partial y}}\right]$$

Now x, y and z are perfect differential

$$\therefore \frac{\partial^2}{\partial x\partial z} = \frac{\partial^2}{\partial z\partial x};\quad \frac{\partial^2}{\partial y\partial z} = \frac{d^2}{\partial z\partial y}$$

and $\dfrac{\partial^2}{\partial x\partial y} = \dfrac{\partial^2}{\partial y\partial x}$.

(38)

$$\therefore L_x L_y = -\hbar^2 \left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right]$$

$$= i\hbar L_z$$

$$\therefore [L_x, L_y] = i\hbar L_z$$

Similarly $[L_y, L_z] = i\hbar L_x$ —— (C)

and $[L_z, L_x] = i\hbar L_y$

(D) commutation relation of L^2 with L_x, L_y & L_z:-

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}_x^2 \hat{L}_x] + [L_y^2, L_x] + [\hat{L}_z^2 \hat{L}_x] \text{ —— (1)}$$

But $\hat{L}_x^2, \hat{L}_x = \hat{L}_x \hat{L}_x, \hat{L}_x$

$$= L_x [L_x, L_x] + [L_x, L_x] L_x = 0 \text{ —— (2)}$$

$$\boxed{\because [A, B] = (-B, A)}$$

$$L_y^2, \hat{L}_x = [\hat{L}_y \hat{L}_y, \hat{L}_x]$$

$$= L_y [L_y, L_x] + [L_y L_x] L_y \text{ —— (3)}$$

$$= -i\hbar L_z \cdot L_y + (-i\hbar L_z L_y)$$

And $[\hat{L}_z^2 \hat{L}_x] = \hat{L}_z [\hat{L}_z \cdot \hat{L}_x] + [\hat{L}_z \hat{L}_x] \hat{L}_z$

$$= L_z \, i\hbar L_y + i\hbar L_y L_z \text{ —— (4)}$$

Putting the value of (2), (3), (4) in (1) we get

$$[L^2, L_x] = i\hbar [-\hat{L}_z \cdot \hat{L}_y - \hat{L}_z \hat{L}_y + \hat{L}_z \hat{L}_y + \hat{L}_y \hat{L}_z]$$

$$\ominus$$

Similarly $[\hat{L}^2, \hat{L}_y] = 0$ —— (5)

& $[\hat{L}^2 \; L_z] = 0$

Eigen value (eigen function) of $\hat{L}_z$

The eigen value equation of $\hat{L}_x$ is given by

$$\hat{L}_z \psi = C\psi$$

Where ψ is the eigen function and c is the eigen value. In spherical co-ordinate

$$-i\hbar \frac{\partial \psi}{\partial \phi} = C\psi$$

The solution of this equation is

$$\psi = f(r,\theta)\, e^{iC\psi/\hbar}$$

where $f(r,\theta)$ is an arbitrary function of r and θ. As ψ is single function it should not be change if ϕ is increased by 2π i.e

$$f(r,\theta)\, e^{iC\psi/\hbar} = f(r,\theta)\, e^{iC(\phi+2\pi)/\hbar}$$

$$\therefore e^{2\pi i c/\hbar} = 1$$

$$\text{or, } \frac{2\pi C}{\hbar} = 2m\pi$$

Where $m = 0, 1, 2, 3 \ldots$ etc.

Thus eigen value of L_z are

$$\hat{L}_z = m\hbar$$

and eigen function are

$$\psi = e^{im\phi} f(r,\theta)$$

Eigen values of $\hat{L}^2$:-

The term $\hat{L}^2$ commutes with L_z, the eigen function of L_z will also be an eigen function of L^2. So it is clear that the repeated application of $L_x + i L_y$ will generate eigen function of L^2. Similarly repeated application of $L_x + i L_y$ will generate eigen function of L_z unless there is some value of $m\ (= m_2)$ for which $(L_x + iL_y) \times \psi_{m_2}$ vanishes.

$$\left.\begin{aligned} \therefore (L_x + iL_y)\psi_{m_1} &= 0 \\ \text{and} \quad (L_x + iL_y)\psi_{m_2} &= 0 \end{aligned}\right\} \text{——— } A$$

Where m_1 is the max^m positive value of $L_z/\hbar$ & m_2 is the maximum negative value.

$$\text{Now} \quad \hat{L}^2\psi_{m_1} = (\hat{L}_x^2 + L_y^2 + L_z^2)\psi_{m_1}$$

$$= \left[(L_x + iL_y)(L_x - iL_y) - i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x) + L_z^2\right]\psi_{m_1}$$

$$= \left[(L_x + iL_y)(L_x - iL_y) + \hbar \hat{L}_z + L_z^2\right]\psi_{m_1}$$

$$= (L_x + iL_y)(L_x - iL_y)\psi_{m_1} + \hbar m_1 \hbar \psi_{m_1} + \hbar^2 m_1^2 \psi_{m_1}$$

$$= (L_x + iL_y)(L_x - iL_y)\psi_{m_1} + \hbar^2(m_1^2 + m_1)\psi_{m_1}$$

$$\left[\because L_z\psi_{m_1} = m_1\hbar\psi_{m_1}\right]$$

$$\text{or,} \quad \hat{L}^2\psi_{m_1} = \hbar^2 m_1(m_1+1)\psi_{m_1} \quad \text{——— } (2) \qquad \& \ (L_x + iL_y)\psi_{m_1} = 0$$

$$\text{Similarly} \quad \hat{L}^2\psi_{m_2} = \hbar^2 m_2(m_2+1)\psi_{m_2} \quad \text{——— } (3)$$

If the equⁿ 2 & 3 are simultanously then

$$m_2(m_2-1) = m_1(m_1+1)$$

or, $m_2^2 - m_2 = m_1^2 + m_1$

or, $m_2^2 - m_1^2 = m_1 + m_2$

or, $(m_2+m_1)(m_2-m_1) = (m_1+m_2)$

$$\Rightarrow (m_2+m_1)(m_2-m_1-1) = 0$$

$$\left.\begin{array}{l}\Rightarrow m_2 = -m_1 \\ \text{or } m_2 = m_1+1\end{array}\right\} \quad (4)$$

Now $m_2 = m_1+1$ is inadmissible by hypotheses m_1 is the largest possible value $L_z/\hbar$

$$\therefore m_2 = -m_1 = -l \text{ (say)}.$$

From equⁿ 2 & 3

$$L^2 \psi_{m_1} = \hbar^2 l(l+1) \psi_{m_1}$$

or, $\hat{L}^2 = \hbar^2 l(l+1)$

This is eigen value of L^2.