Dr Mirza Abbas Hussain
Deptt of Physics, oriental college. B.Sc Paper III Gr B.

Scattering of electromagnetic wave.

(a) Scattering by free charge or Thomson's scattering

Let us consider an electron of mass m and charge "e" in the path of Polarised light.

Let the incident E.M Wave having electric intensity E along x-direction moving along z-direction. The electromagnetic force is $e.E$.

The equation of motion of electron is given by

$$m.\frac{d^2x}{dt^2} = eE = eE_0 e^{-i(\omega t - Kx)} \qquad \text{(A)}$$

$$\text{or, } \frac{d^2x}{dt^2} = \frac{eE_0}{m} e^{-i(\omega t - Kx)} \qquad \text{(B)}$$

$$\Rightarrow x = -\frac{eE_0}{\omega^2 m} e^{-i(\omega t - Kx)}$$

The oscillating charge behaves like a dipole having dipole moment

$$p = ex = -\frac{e^2 E_0}{\omega^2 m} e^{-i(\omega t - Kx)}$$

or

$$p = p_0 e^{-i(\omega t - Kx)}$$

$$\text{where } p_0 = -\frac{e^2 E_0}{m\omega^2}$$

The oscillating dipole radiates energy. The average energy radiating per second per unit area is given by

$$\omega_s = \frac{1}{4\pi\epsilon_0} \frac{\omega^4 p_0^2}{8\pi c^3 r^2} \sin^2\alpha.$$

$$= \frac{1}{4\pi\epsilon_0} \frac{e^4 E_0^2}{8\pi c^3 r^2 m^2} \sin^2\alpha \qquad \text{(C)}$$

For plane electromagnetic wave the average incident energy is given by

$$\omega_i = \frac{1}{2} c\epsilon_0 E_0^2$$

The differential scattering cross-section is given by

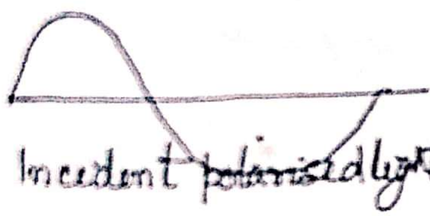
$$\sigma(\theta) = \frac{\omega_s}{\omega_i} r^2$$

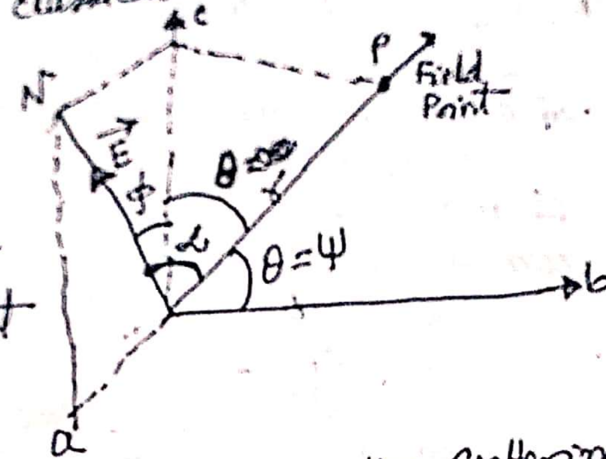
$$= \frac{1}{4\pi\epsilon_0} \frac{e^4 E_0^2 \sin^2\alpha}{8\pi c^3 r^2 m^2} \cdot \frac{2}{c\epsilon_0 E_0^2} r^2$$

or, $\sigma(\alpha) = \left(\frac{e^2}{4\pi mc^2\epsilon_0}\right)^2 \sin^2\alpha$ (2)

or, $\sigma(\theta) = r_0^2 \sin^2\alpha$.

where $r_0 = \frac{e^2}{4\pi mc^2\epsilon_0}$ = classical radius of the electron.





Let ϕ is the polaring angle and θ is the scattering angle of e.m. wave and α is the angle between position vector r & electric vector E

Now $ON = r\cos\alpha$ & $OM = r\sin\theta$

$\therefore ON = r\cos\alpha = OM\cos\phi = r\sin\theta\cos\phi$.

Again $\cos^2\alpha = \sin^2\theta \cdot \cos^2\phi$.

$\therefore 1 - \sin^2\alpha = (1-\cos^2\theta)\cos^2\phi$.

or, $\sin^2\alpha = 1 - \cos^2\phi(1-\cos^2\theta)$

For plane polarised light, $\phi = 0$

$\therefore \sin^2\alpha = \cos^2\theta$

For unpolarised light, the average value of ϕ is equal to $\sqrt{1/2}$.

$\therefore \sin^2\alpha = 1 - \frac{1}{2}(1-\cos^2\theta) = \frac{1}{2}(1+\cos^2\theta)$

$= \frac{1}{2}(1+\cos^2\theta)$

The scattering cross-section area is given by

$$\boxed{\sigma(\theta) = r_0^2 \frac{1}{2}(1+\cos^2\theta)}$$

This is thomson scattering formula.

⑧ Scattering by bounded charge (Rayleigh Scattering)

Let us consider following forces are acting upon the bounded electrons.

(i) Inertial force $= m \cdot \frac{d^2x}{dt^2}$

(ii) Frictional force which is proportional to velocity
$$f_r = -\alpha \frac{dx}{dt}$$

(iii) Damping force which is proportional to the displacement $f_t = Kx$

(iv) The periodic electromagnetic force.
$$F = eE_0 e^{i(\omega t - Kx)}.$$

The eqn of motion is given by.
$$m \cdot \frac{d^2x}{dt^2} + \alpha \cdot \frac{dx}{dt} + Kx = eE_0 e^{-i(\omega t - Kx)}$$

$$\text{or, } \frac{d^2x}{dt^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{K}{m}x = \frac{e}{m}E_0 e^{-i(\omega t - Kx)} \quad \text{(A)}$$

$$\text{or, } \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{e}{m}E_0 e^{-i(\omega t - Kx)}$$

where $\gamma = \alpha/m$ & $\omega_0^2 = K/m$.

The solution of complementary function is given by.
$$\frac{d^2x}{dt^2} + \gamma \cdot \frac{dx}{dt} + \omega_0^2 x = 0 \quad \text{(B)}$$

Let $x = Ae^{\alpha t}$, the eqn 'B' may be written as
$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0$$
$$\therefore \alpha = -\frac{\gamma}{2} \pm \sqrt{\gamma^2/4 - \omega_0^2}$$
$$= -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \gamma^2/4}$$
$$\therefore x = e^{-\gamma t/2}\left[A_1 e^{+i\sqrt{(\omega^2 + \gamma^2/4)}\,t} + A_2 e^{-\left(i\sqrt{\omega^2 - \gamma^2/4}\right)t}\right] \quad \text{(C)}$$

The factor decreases exponentially which die after some time. The solⁿ of Perpendicular integral is given by
$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2(x) = e/m\, E_0 e^{-i(\omega t - Kx)} \quad \text{(D)}$$

Let $x = B e^{-i(\omega t - Kx)}$

from eqn (2) we get

$$B = \frac{eE_0}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

$$\therefore x = \frac{eE_0}{m(\omega_0^2 - \omega^2 - i\gamma\omega)} e^{-i(\omega t - Kx)}$$

$$= \frac{eE_0[(\omega_0^2 - \omega^2) + i\gamma\omega]}{m[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]} e^{-i(\omega t - Kx)}$$

$$= \frac{eE_0}{m} \frac{e^{-i(\omega t - Kx)}}{[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}} \left[\frac{\omega_0^2 - \omega^2}{[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}} + \frac{i\gamma\omega}{[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}}\right]$$

$$= \frac{eE_0}{m} \frac{e^{-i(\omega t - Kx)}}{[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}} [\cos\theta - i\sin\theta)$$

$$= \frac{eE_0}{m} \frac{e^{-i(\omega t - Kx) + \delta)}}{[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}} \quad \text{—— (E)}$$

where $\delta = \tan^{-1} \frac{\gamma\omega}{\omega_0^2 - \omega^2}$

The comple solution of equation (A) is given by

$$x = e^{-\gamma t/2}[A_1 e^{i\beta t} + A_2 e^{-i\beta t}] + \frac{eE_0}{m} \frac{e^{i(\omega t - Kx + \delta)}}{[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}} \quad \text{—— (F)}$$

The first term may be neglected due to -ve exponential

The general eqn of oscillating charge due to dipole moment

$$p = ex$$

$$= \frac{e^2 E_0}{m[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}} \left[e^{+i(\omega t - Kx + \delta}\right].$$

$$\text{or, } p = p_0 e^{-i(\omega t - Kx - \delta)} \quad \text{—— (G)}$$

$$\text{where } p_0 = \frac{e^2 E}{m[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}} \quad \text{—— (H)}$$

The average energy radiated per second per unit area is given by.

(9')

$$S = \frac{1}{4\pi\epsilon_0}\frac{\omega^4}{8\pi c^2 r^2}\cdot p_0^2 \sin^2\theta.$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{\omega^4 p^2}{8\pi c^2 r^2}\,\frac{e^4 E_0^2 \sin^2\theta}{m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]} \quad \text{—— (H)}$$

The average incident radiation (Poynting Vector)

$$S_{in} = \frac{1}{2}\epsilon_0 c E_0^2$$

The differential scattering crossection is given by

$$\sigma(\theta) = \frac{S}{S_{in}} \cancel{\gamma^2}\, \gamma^2$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{\omega^4}{8\pi c^2 r^2}\,\frac{e^2 E_0^2 \sin^2\theta\, r^2}{m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]\, c\epsilon_0 E_0^2} \quad \text{—— (I)}$$

On solving we get

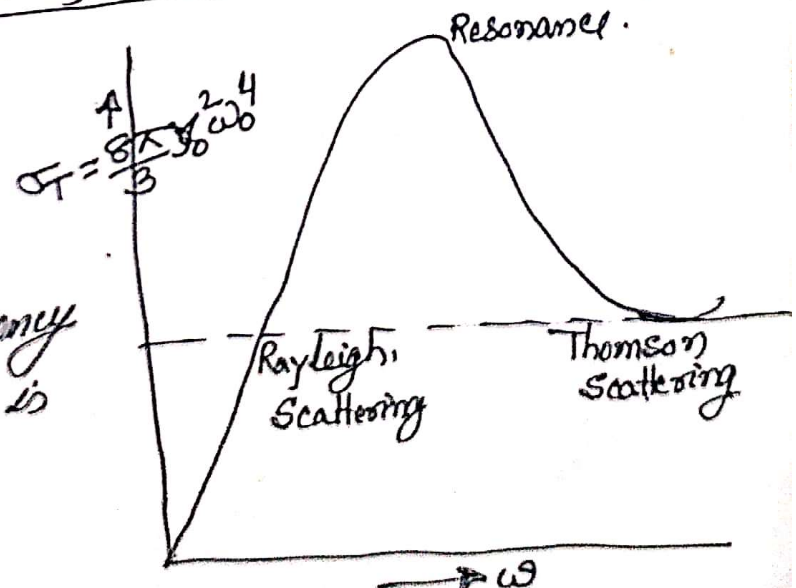
$$\sigma(\theta) = \gamma_0^2\omega^4\left(\frac{1+\cos^2\theta}{2}\right)\Big/(\omega_0^2-\omega^2)^2+\gamma^2\omega^2$$

The total scattering crossection is given by

$$\sigma_T = \int_0^\pi \frac{\gamma_0^2\omega^4}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}\left(\frac{1+\cos^2\theta}{2}\right)\cdot 2\pi\sin\theta\, d\theta \quad \text{—— J}$$

$$\boxed{\sigma_T = \frac{8\pi}{3}\frac{\gamma_0^2\omega^4}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}}$$

The eqn I represents the total scattering crossection for elastically bound electron. The scattering crossection is a function of frequency of incident radiation is given by Graph A.



Gr A